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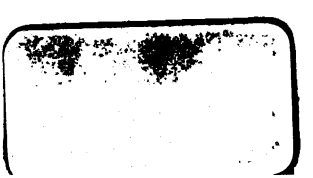

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ALGEBRA.

PART II

BY

THE REV. T. DALTON, M.A.

ASSISTANT MASTER AT ETON COLLEGE.

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ALGEBRA.

CHAPTER I.

Numerical Substitution, and Explanation of Terms.

1. The student has already had considerable practice in examples in easy numerical substitution: he may now with advantage give his attention to some which are less simple and involve the use of other symbols.

2. It has been explained that the square root of any number is that number which when multiplied by itself produces the given number. The symbol $\surd$ (which may have originally been the first letter of the word *radix* or *root*) is used to denote 'square root': thus

$\sqrt{25}$ signifies the square root of 25, i.e. 5,

$\sqrt{9a^2b^6}$ ..... $9a^2b^6$, i.e. $3ab^3$.

Similarly the cube root of any number is that number which when multiplied into itself three times produces the given number: thus, $125 = 5 \times 5 \times 5$, and therefore 5 is the cube root of 125; or again

$$64x^3y^6z^9 = 4xy^2z^3 \times 4xy^2z^3 \times 4xy^2z^3,$$

and therefore $4xy^2z^3$ is the cube root of $64x^3y^6z^9$. The symbol $\sqrt[3]{\ }$ is used to denote 'cube root', so that we can briefly write

$$\sqrt[3]{125} = 5; \quad \sqrt[3]{64x^3y^6z^9} = 4xy^2z^3.$$

In the same way, the fourth, fifth, or any higher root of a number denotes the number which when multiplied into itself four, five,..... times produces the given number; and the symbols $\sqrt[4]{}$, $\sqrt[5]{}$, $\sqrt[6]{}$,..... are used to denote these roots: thus

$$\sqrt[4]{625}=5;\quad \sqrt[5]{32a^{10}}=2a^2;\quad \sqrt[6]{729x^6}=3x.$$

3. The term *power* is the converse of root, and denotes the number of times a quantity is multiplied into itself to make it equal to some other quantity: thus

125 (or 5^3) is the third power of 5,

$81x^4$ (or $3x)^4$ ..... fourth ..... of $3x$,

$243a^{10}b^{15}$ (or $3a^2b^3)^5$ ..... fifth ..... of $3a^2b^3$,

$64a^{12}$ (or $2a^2)^6$ ..... sixth ..... of $2a^2$.

4. The name *expression* is given to any collection of algebraical symbols: it generally consists of *terms* connected by the signs + or −; if there be only one term in it, it is called a *simple expression;* if more than one, it is a *compound expression:* thus

$20a^2bx^5$ is a simple expression,

$2a^2+5ab-bc$ is a compound expression.

A compound expression is said to be *binomial* if it consist of two terms, *trinomial* if of three terms, *multinomial* or *polynomial* if it have four or more than four terms.

5. The *dimensions* of any algebraical quantity are determined by taking the sum of the indices of all its letters; thus $7a^2b^3$ is of five dimensions, $16x^2yz^6$ is of nine dimensions.

Homogeneous is the designation applied to any compound expression whose terms are all of the same dimen-

sions: thus $5a^3b^2-a^2b^3c+7abcde-2b^5$ is homogeneous, because all its terms are of five dimensions.

6. In any expression, *like* terms are those which differ only in their numerical coefficients; *unlike* terms may differ either in their letters or in the indices of the same letter: thus x, $6x$, $-2x$ are like terms, but x^2, $2x$, xy, $2y$ are all unlike.

7. The *factors* of any term are the quantities which compose it by multiplication; thus in $15xy$ the factors are 15, x, and y: in $7a(b-c)(c-d)$ the factors are 7, a, $b-c$, and $c-d$.

When an expression contains brackets with no sign between them, the sign of multiplication is always understood.

One or more of the factors of any term may be considered as the coefficient of the rest, but this designation is in practice almost entirely restricted to the *numerical* factor of the term.

8. Exs. If $a=6$, $b=4$, $c=3$, $d=0$, $e=-1$, $f=\cdot2$, find the numerical value of

(i) $\sqrt{3a+2c^2}-\sqrt[3]{b^2+c^2+2}$;

(ii) $5(a-b)(c^2+bd-e^3)-2(a+e)^2$;

(iii) $10(cf+ad)-(2b-c)(a+e)+\sqrt[5]{5f+bc+2c^2}$.

(i) $\sqrt{3a+2c^2}-\sqrt[3]{b^2+c^2+2}$

$=\sqrt{18+18}-\sqrt[3]{16+9+2}=\sqrt{36}-\sqrt[3]{27}$

$=6-3=3$. Ans.

(ii) $5(a-b)(c^2+bd+e^2)-2(a+e)^2$

$=5(6-4)(9+0+1)-2(6-1)^2$

$=5\times2\times10-2\times25=100-50=50$. Ans.

(iii) $10(cf+ad)-(2b-c)(a+e)+\sqrt[5]{10f+bc+2c^3}$

$=10\left(\frac{3}{5}+0\right)-(8-3)(6-1)+\sqrt[5]{2+12+18}$

$=6-5\times 5+\sqrt[5]{32}$

$=6-25+2$

$=-17.$ Ans.

In an example like this, where the value of one of the letters is a decimal fraction, the decimal form may be retained, but it is generally better to convert it at once into a vulgar fraction.

Exercise I.

If $a=0$, $b=1$, $c=2$, $d=3$, $x=4$, $y=5$, $z=6$, find the numerical value of

1. $\sqrt{2bc}$.
2. $\sqrt{4cdz}$.
3. $\sqrt{2cd^2y^2}$.
4. $\sqrt{3cd^2z}$.
5. $\sqrt{10b^2dyz}$.
6. $\sqrt{12c^2dy^2}$.
7. $\sqrt{\frac{c^3z}{d}}$.
8. $\sqrt{\frac{3cxy}{5z}}$.
9. $\sqrt{\frac{2d^2ryz^2}{5c^3}}$.
10. $\sqrt{4d+xz}$.
11. $\sqrt{6c^2+4b^2}$.
12. $\sqrt{12xy-11bc^2}$.
13. $\sqrt{a^2+d^2+x^2}$.
14. $\sqrt{3by+2dx-3b^2}$.
15. $\sqrt{3ax+5by+2cz}$.
16. $\sqrt{\frac{2dx-3cy+yz}{ab+cd}}$.
17. $\sqrt{\frac{2y^2+3x^2-5a^2}{3d^2-y^2}}$.
18. $\sqrt{\frac{2(ab+cd+xy)}{c^2x-b^2d}}$.
19. $\sqrt{\frac{3ac^2+2xy^2-6z^3}{2(2c^3+4cd^2-d^2y^2)}}$.
20. $\sqrt{\frac{x^2y-cdz+3b^2c}{z^2-2y^2+x^2}}$.

21. $\sqrt[3]{2cx^2}$. 22. $\sqrt[3]{2cd^2z}$. 23. $\sqrt[3]{-15b^2d^2y^5}$.

24. $\sqrt[3]{-12c^3dz}$. 25. $\sqrt[3]{3ayz^2+2c^2y^2+4b^3x}$.

26. $\sqrt[3]{2x^2-3y^2+7by}$. 27. $\sqrt[4]{y^3-d^2}$.

28. $\sqrt[4]{3cy^2-x^3-5b^3}$. 29. $\sqrt[5]{12d^2-3x^3-2y^4-9b^3}$.

30. $\sqrt[5]{d^4-c^3x^3+b^2dy}$. 31. $(a+b+c)(d+x)$.

32. $(b+c)(d+x+y)$. 33. $(z-y)(x+d-c)$.

34. $3(a+d)(c+y)$. 35. $7(z-y)(b-c+d)$.

36. $2(c^2+d^2)(a^2+b^2)$. 37. $3(a+d)+(c+y)$.

38. $7(z-y)-(b-c+d)$.

39. $4(x^2-a^2)-2(y^2-c^2)-3(z^2-d^2)$.

40. $4(x-a)^2-2(y-c)^2-3(z-d)^2$.

41. $(2b+c)^2-2(b+c)^2+2(b^2+c^2)$.

42. $(3x-2y)^2-3(2x-y)^2+2(x^2-y^2)$.

43. $(c+d)^3+4(d-c)^3-(4d-3y)^3$.

44. $(xy-yz)^2-3(2dx-y^2)^2$.

45. $(a-b+c)^2-3(b-c+d)^2+x(x-y+z)^2$.

46. $a(b+c)^2+b(c+d)^2$.

47. $a^2(b^2+c^2)+b^2(a^2+c^2)+c^2(a^2+b^2)$.

48. $x^2(x-y)^2-y^2(y-z)^2+z^2(z-x)^2$.

49. $(3a+b)(3a^2-b^2)-(2x+y)(2x^2-y^2)$.

50. $4(b+c)(b-c)^2-3(x+y)(x-y)^2$.

If $a=10$, $b=2$, $c=0$, $k=\cdot4$, $l=-5$, find the numerical value of

51. $ak+bc+bl$. 52. $3bk-2cl+abl$.

53. $a^2+c^2+k^2+l^2$. 54. $5b^2c+ak^2-2b^2l$.

55. $\sqrt{-bkl}$. 56. $\sqrt{b^2+k^2-2bk}$. 57. $\sqrt[3]{a^2+l^2}$.

58. $a-(\sqrt{abk+b^3}+b\sqrt{ab-l})$.

59. $\dfrac{3\sqrt{l^2+a+1}-2\sqrt{2ak}}{l^2(b+k)^2}$.

60. $\dfrac{3(a^2-bc)-10(k^2+2kc+c^2)}{\sqrt{a(b+4k)}-\sqrt[5]{3b^3+2kl+5ac}}$.

CHAPTER II.

FRACTIONAL AND NEGATIVE INDICES.

9. WE have hitherto dealt with letters having only positive integers as their indices, but in exactly the same way they may have for their indices fractions or negative quantities. This subject will be more fully explained in Chapter XXIV; at present it is sufficient to state that the interpretation of quantities with fractional or negative indices is determined subject to the condition that the rules for their multiplication and division are the same as when the indices are positive integers; viz. that in multiplication we take the *sum* of the indices, and in division we take their *difference:* thus

$$a^{\frac{2}{5}}\times a^{\frac{1}{5}}=a^{\frac{3}{5}};\quad 2a^{\frac{1}{2}}\times 5a^{\frac{1}{2}}=10a^{\frac{5}{6}};$$

$$4x^{\frac{1}{5}}yz^{\frac{3}{4}}\times 3x^{\frac{2}{5}}y^{\frac{1}{4}}z^{\frac{1}{3}}=12x^{\frac{3}{5}}y^{\frac{5}{4}}z;$$

$$x^{-5}\times 2x^{-1}=2x^{-6};\quad 3a^4b^{-2}\times 5a^{-3}b^{-1}=15ab^{-3}.$$

10. From the above condition it may be shewn that a fractional index indicates a *root* of the letter to which it is attached: thus $x^{\frac{1}{2}}$ is the same as $\sqrt{x}$; $a^{\frac{1}{3}}$ is identical with $\sqrt[3]{a}$; $a^{\frac{2}{5}}=\sqrt[5]{a^2}$. A negative index indicates that the letter

to which it is attached is the *denominator* of a fraction; or if it be already in the denominator, that it is to be written in the numerator: thus

$$a^{-2}=\frac{1}{a^2};\quad x^{-\frac{1}{3}}=\frac{1}{x^{\frac{1}{3}}}=\frac{1}{\sqrt[3]{x}};\quad x^{-\frac{5}{6}}=\frac{1}{x^{\frac{5}{6}}}=\frac{1}{\sqrt[6]{x^5}};\quad \frac{1}{a^{-6}}=a^6;$$

$$\frac{3}{a^{-\frac{5}{2}}}=3a^{\frac{5}{2}}=3\sqrt{a^5}.$$

11. The student must carefully note that $x^0=1$; this result may be obtained as follows:

$$x^a\times x^{-a}=x^{a-a}=x^0,$$

but again $$x^a\times x^{-a}=x^a\times\frac{1}{x^a}=1,$$

hence $$x^0=1.$$

12. Ex. 1. Multiply $4x^{\frac{5}{6}}+3x^{\frac{2}{3}}+x^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-2x^{\frac{1}{3}}$.

$$\begin{array}{l} 4x^{\frac{5}{6}}+3x^{\frac{2}{3}}+\ x^{\frac{1}{2}} \\ x^{\frac{1}{2}}-2x^{\frac{1}{3}} \\ \hline 4x^{\frac{4}{3}}+3x^{\frac{7}{6}}+\ x \\ \qquad -8x^{\frac{7}{6}}-6x\ -2x^{\frac{5}{6}} \\ \hline 4x^{\frac{4}{3}}-5x^{\frac{7}{6}}-5x\ -2x^{\frac{5}{6}}. \quad \text{Ans.} \end{array}$$

Ex. 2. Multiply $3x-2+x^{-1}-2x^{-2}$ by $x+3+2x^{-1}$.

$$\begin{array}{llll} 3x & -2 & +x^{-1}-2x^{-2} & \\ x & +3 & +2x^{-1} & \\ \hline 3x^2-2x+1 & & -2x^{-1} & \\ \qquad 9x-6 & & +3x^{-1}-6x^{-2} & \\ \qquad\quad 6 & & -4x^{-1}+2x^{-2}-4x^{-3} & \\ \hline 3x^2+7x+1 & & -3x^{-1}-4x^{-2}-4x^{-3}. & \text{Ans.} \end{array}$$

Ex. 3. Multiply $x^{\frac{5}{6}}y^{-1}+2x^{-\frac{1}{6}}y^{-2}-x^{-\frac{7}{6}}y^{-3}$ by $2x^{\frac{1}{6}}-4x^{-\frac{5}{6}}y^{-1}$.

$$
\begin{array}{llll}
x^{\frac{5}{6}}y^{-1}+2x^{-\frac{1}{6}}y^{-2}- & x^{-\frac{7}{6}}y^{-3} & & \\
2x^{\frac{1}{6}} & -4x^{-\frac{5}{6}}y^{-1} & & \\
\hline
2xy^{-1} & +4y^{-2} & -x^{-1}y^{-3} & \\
 & -4y^{-2} & -8x^{-1}y^{-3}+4x^{-2}y^{-4} & \\
\hline
2xy^{-1} & & -9x^{-1}y^{-3}+4x^{-2}y^{-4}. & \text{Ans.}
\end{array}
$$

13. In Division, if the fractional indices of the same letter have different denominators, the work will be simplified by first writing them with a common denominator: thus

$$x^{\frac{1}{2}}\div x^{\frac{1}{3}}=x^{\frac{3}{6}}\div x^{\frac{2}{6}}=x^{\frac{1}{6}};$$

$$4x^{-2}y^{\frac{5}{6}}z^{\frac{1}{2}}\div 2x^{-3}y^{\frac{1}{3}}z=4x^{-2}y^{\frac{5}{6}}z^{\frac{1}{2}}\div 2x^{-3}y^{\frac{1}{3}}z^{\frac{2}{2}}=2xy^{\frac{1}{6}}z^{-\frac{1}{2}}.$$

Ex. 4. Divide $64x^{\frac{9}{10}}-27y$ by $4x^{\frac{3}{10}}-3y^{\frac{1}{3}}$.

$$
\begin{array}{l|l}
4x^{\frac{3}{10}}-3y^{\frac{1}{3}} & 64x^{\frac{9}{10}}-27y^{\frac{3}{3}} \quad (\ 16x^{\frac{6}{10}}+12x^{\frac{3}{10}}y^{\frac{1}{3}}+9y^{\frac{2}{3}} \\
 & \underline{64x^{\frac{9}{10}}-48x^{\frac{6}{10}}y^{\frac{1}{3}}} \\
 & \quad 48x^{\frac{6}{10}}y^{\frac{1}{3}}-27y^{\frac{3}{3}} \\
 & \quad \underline{48x^{\frac{6}{10}}y^{\frac{1}{3}}-36x^{\frac{3}{10}}y^{\frac{2}{3}}} \\
 & \qquad 36x^{\frac{3}{10}}y^{\frac{2}{3}}-27y^{\frac{3}{3}} \\
 & \qquad \underline{36x^{\frac{3}{10}}y^{\frac{2}{3}}-27y^{\frac{3}{3}}}
\end{array}
$$

Hence the quotient is $16x^{\frac{6}{5}}+12x^{\frac{3}{5}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}$.

Ex. 5. Divide $12xy^{\frac{1}{2}}+5x^{\frac{2}{3}}y^{\frac{1}{2}}-12x^{\frac{1}{3}}y^{\frac{2}{3}}+y+x^{-\frac{1}{2}}y^{\frac{5}{4}}-2x^{-\frac{2}{3}}y^{\frac{2}{3}}$ by $4x^{\frac{1}{2}}-y^{\frac{1}{2}}-2x^{-\frac{1}{2}}y^{\frac{1}{3}}$.

$4x^{\frac{1}{3}} - y^{\frac{1}{4}} - 2x^{-\frac{1}{3}}y^{\frac{2}{4}}$)

$12x^{\frac{2}{3}}y^{\frac{1}{4}} + 5x^{\frac{2}{3}}y^{\frac{2}{4}} - 12x^{\frac{1}{3}}y^{\frac{3}{4}} + y^{\frac{4}{4}} + x^{-\frac{1}{3}}y^{\frac{5}{4}} - 2x^{-\frac{2}{3}}y^{\frac{6}{4}}$ ($3x^{\frac{2}{3}}y^{\frac{1}{4}} + 2x^{\frac{1}{3}}y^{\frac{2}{4}} - y^{\frac{3}{4}} + x^{-\frac{1}{3}}y^{\frac{4}{4}}$

$12x^{\frac{2}{3}}y^{\frac{1}{4}} - 3x^{\frac{2}{3}}y^{\frac{2}{4}} - 6x^{\frac{1}{3}}y^{\frac{3}{4}}$

$8x^{\frac{2}{3}}y^{\frac{2}{4}} - 6x^{\frac{1}{3}}y^{\frac{3}{4}} + y^{\frac{4}{4}}$

$8x^{\frac{2}{3}}y^{\frac{2}{4}} - 2x^{\frac{1}{3}}y^{\frac{3}{4}} - 4y^{\frac{4}{4}}$

$- 4x^{\frac{1}{3}}y^{\frac{3}{4}} + 5y^{\frac{4}{4}} + x^{-\frac{1}{3}}y^{\frac{5}{4}}$

$- 4x^{\frac{1}{3}}y^{\frac{3}{4}} + y^{\frac{4}{4}} + 2x^{-\frac{1}{3}}y^{\frac{5}{4}}$

$4y^{\frac{4}{4}} - x^{-\frac{1}{3}}y^{\frac{5}{4}} - 2x^{-\frac{2}{3}}y^{\frac{6}{4}}$

$4y^{\frac{4}{4}} - x^{-\frac{1}{3}}y^{\frac{5}{4}} - 2x^{-\frac{2}{3}}y^{\frac{6}{4}}$

Hence the quotient is $3x^{\frac{2}{3}}y^{\frac{1}{4}} + 2x^{\frac{1}{3}}y^{\frac{2}{4}} - y^{\frac{3}{4}} + x^{-\frac{1}{3}}y$.

Ex. 6. Extract the square root of

$$4x^{\frac{3}{2}} - 12x^{\frac{3}{4}}y^{-\frac{1}{2}} + 29y^{-1} - 30x^{-\frac{3}{4}}y^{-\frac{3}{2}} + 25x^{-\frac{3}{2}}y^{-3}.$$

$4x^{\frac{3}{4}} - 12x^{\frac{3}{4}}y^{-\frac{1}{2}} + 29y^{-\frac{3}{2}} - 30x^{-\frac{3}{4}}y^{-\frac{3}{2}} + 25x^{-\frac{6}{4}}y^{-\frac{4}{2}}$ ($2x^{\frac{3}{4}} - 3y^{-\frac{1}{2}} + 5x^{-\frac{3}{4}}y^{-\frac{3}{2}}$

$4x^{\frac{6}{4}}$

$4x^{\frac{3}{4}} - 3y^{-\frac{1}{2}}$ | $- 12x^{\frac{3}{4}}y^{-\frac{1}{2}} + 29y^{-\frac{3}{2}}$

$- 12x^{\frac{3}{4}}y^{-\frac{1}{2}} + 9y^{-\frac{3}{2}}$

$4x^{\frac{3}{4}} - 6y^{-\frac{1}{2}} + 5x^{-\frac{3}{4}}y^{-\frac{3}{2}}$ | $20y^{-\frac{3}{2}} - 30x^{-\frac{3}{4}}y^{-\frac{3}{2}} + 25x^{-\frac{6}{4}}y^{-\frac{4}{2}}$

$20y^{-\frac{3}{2}} - 30x^{-\frac{3}{4}}y^{-\frac{3}{2}} + 25x^{-\frac{6}{4}}y^{-\frac{4}{2}}$

The square root is $2x^{\frac{3}{4}} - 3y^{-\frac{1}{2}} - 5x^{-\frac{3}{4}}y^{-1}$. Ans.

Exercise II.

Multiply

1. $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$. 2. $4x^{\frac{2}{5}}$ by $x^{1\frac{3}{5}}$. 3. $2a^{\frac{1}{2}}b^{\frac{2}{5}}$ by $7a^{\frac{1}{3}}b^{1\frac{3}{5}}$.

4. $4a^{\frac{3}{4}}b^{\frac{2}{5}}$ by $12a^{\frac{1}{4}}b^{\frac{1}{5}}$. 5. $15a^{\frac{3}{4}}b^{\frac{1}{5}}c^{\frac{5}{12}}$ by $5a^{\frac{1}{2}}b^{1\frac{2}{5}}c^{\frac{1}{4}}$.

6. $3xy^{\frac{2}{3}}$ by $6x^{\frac{2}{5}}y^{\frac{1}{3}}$. 7. $4a^{\frac{2}{3}}bc^{\frac{1}{5}}$ by $8a^{\frac{1}{3}}b^{\frac{2}{5}}c^{\frac{2}{5}}$.

8. $2x^2y^{\frac{5}{6}}z^3$ by $20x^{\frac{1}{3}}y^{\frac{7}{6}}z^{\frac{1}{2}}$. 9. $5a^{\frac{2}{3}}b^{\frac{5}{7}}c^{1\frac{2}{5}}d$ by $a^{\frac{1}{2}}b^3cd^{\frac{1}{5}}$.

10. $3a^3b^{1\frac{5}{12}}d^{\frac{2}{5}}e$ by $7a^{\frac{4}{3}}c^{\frac{2}{5}}d^{\frac{7}{5}}e^{\frac{7}{5}}$. 11. a^{-3} by $2a^{-5}$.

12. $3a^{-7}b^{-1}$ by $2a^{-2}b^{-4}$. 13. $4a^{-1}b^{-3}$ by $a^{-3}b^2$.

14. $5x^{-6}y^3z^{-7}$ by $3x^4y^{-2}z^{-1}$. 15. $x^7y^{-9}z^2$ by $x^{-6}y^3z^{-2}$.

16. $a^{-\frac{2}{5}}b^{-\frac{3}{4}}$ by $a^{-\frac{3}{5}}b^{-\frac{5}{4}}$. 17. $4a^{-1}b^{-7}c^{-\frac{3}{5}}$ by $a^6b^{\frac{3}{5}}c^{-\frac{2}{5}}$.

18. $3a^{-\frac{4}{9}}b^{\frac{6}{7}}c^{-\frac{4}{5}}$ by $2a^{-\frac{5}{9}}b^{-\frac{6}{7}}c^{-\frac{6}{5}}$.

19. $3x^{\frac{4}{9}}y^{-\frac{7}{5}}z^{-\frac{4}{5}}$ by $2x^{-\frac{1}{3}}y^{-\frac{2}{9}}z^{\frac{4}{5}}$.

20. $7a^{-5}b^{\frac{3}{5}}c^{-\frac{3}{5}}d^{-\frac{4}{5}}$ by $3a^3b^{-\frac{8}{5}}c^{\frac{3}{5}}d^{-\frac{7}{5}}$.

21. $3a^{\frac{5}{6}}+2a^{\frac{1}{2}}$ by $4a^{\frac{1}{2}}$. 22. $9a^{\frac{2}{3}}+3+7a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}$.

23. $9x^2-3-x^{-2}$ by $2x^{\frac{2}{3}}$.

24. $7x^{\frac{6}{5}}+4a^{\frac{1}{2}}x-ax^{\frac{4}{5}}$ by $-11a^{-\frac{1}{2}}x^{-1}$.

25. $x^{\frac{7}{5}}+2x^{\frac{4}{5}}y^{\frac{2}{3}}$ by $x^{\frac{2}{5}}-2y^{\frac{2}{3}}$.

26. $3x^{\frac{7}{5}}-2x^{\frac{7}{6}}-1$ by $x^{\frac{1}{6}}+x^{-1}$.

27. $3a^{\frac{4}{5}}-4a^{\frac{3}{5}}b^{\frac{1}{6}}+5a^{\frac{3}{5}}b^{\frac{1}{2}}$ by $6a^{\frac{6}{5}}+8ab^{\frac{1}{6}}$.

28. $12a^2b^{\frac{3}{5}}-2b+a^{-2}b^{\frac{4}{5}}$ by $2a^{\frac{3}{2}}b^{-\frac{3}{5}}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}$.

29. $4ab^{-2}c^3-3a^{\frac{1}{2}}b^{-1}+5c^{-3}$ by $12a^{\frac{3}{2}}b^3+9ab^2c^{-3}$.

30. $4x^{\frac{5}{2}}y^{-\frac{5}{2}}+8x^{\frac{3}{2}}y^{-\frac{1}{2}}+16x^{\frac{1}{2}}y^{\frac{1}{2}}+32x^{-\frac{1}{2}}y^{\frac{3}{2}}$ by $x^{\frac{3}{2}}y^{-\frac{3}{2}}-2x^{\frac{1}{2}}y^{-\frac{1}{2}}$.

31. $3a^{-4}+4a^{-3}b^{-1}+b^{-2}$ by $4a^{-1}+3ab^{-1}-4a^{2}b^{-2}$.

32. $x^{-\frac{7}{2}}-6x^{-\frac{5}{2}}+3x^{-\frac{3}{2}}$ by $x^{\frac{3}{2}}+4x^{\frac{1}{2}}+2x^{\frac{3}{2}}$.

33. $x^{\frac{5}{2}}+x^{\frac{3}{2}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{-\frac{3}{2}}$ by $x^{\frac{1}{2}}-x^{-\frac{1}{2}}$.

34. $a^{\frac{1}{2}}-2a^{\frac{3}{4}}+4a^{\frac{1}{4}}-8a^{\frac{1}{8}}+16-32a^{-\frac{1}{8}}$ by $a^{\frac{1}{4}}+8a^{-\frac{1}{8}}$.

35. $x^{\frac{3}{2}}-4+4x^{-\frac{3}{2}}$ by $x^{\frac{3}{2}}+4+4x^{-\frac{3}{2}}$.

36. $2a^{-2}-3a^{-1}b^{\frac{1}{2}}+3b$ by $3a^{-2}-4a^{-1}b^{\frac{1}{2}}-2b$.

37. $a^{\frac{5}{6}}+2a^{\frac{3}{6}}b^{\frac{1}{6}}+4a^{\frac{1}{6}}b^{\frac{2}{6}}+8a^{\frac{1}{6}}b^{\frac{3}{6}}$ by $a^{\frac{1}{6}}-4b^{\frac{1}{6}}+4a^{-\frac{1}{6}}b^{\frac{2}{6}}$.

38. $x^{\frac{5}{4}}-2x^{\frac{1}{2}}+3x^{-\frac{1}{4}}$ by $2x^{\frac{1}{2}}-3x^{-\frac{1}{4}}-x^{-1}$.

39. $3a^{-1}y^{\frac{3}{2}}+2a^{-\frac{3}{2}}y^{\frac{1}{2}}-a^{-2}y^{\frac{1}{4}}-2a^{-\frac{5}{2}}$ by $4a-a^{\frac{1}{2}}y^{-\frac{1}{4}}+2y^{-\frac{1}{2}}$.

40. $9a+3a^{\frac{1}{2}}b^{-1}-6a^{\frac{1}{2}}c^{-\frac{1}{6}}+b^{-2}+2b^{-1}c^{-\frac{1}{6}}+4c^{-\frac{2}{6}}$ by $3a^{\frac{1}{2}}-b^{-1}+2c^{-\frac{1}{6}}$.

Divide

41. $x^{\frac{7}{10}}$ by $x^{\frac{1}{5}}$.

42. $12a^{\frac{3}{2}}b^{\frac{3}{2}}$ by $4a^{\frac{1}{2}}b^{\frac{1}{2}}$.

43. $27a^{\frac{4}{3}}b^{\frac{7}{15}}c^{\frac{2}{3}}$ by $9a^{\frac{1}{10}}b^{\frac{1}{3}}c$.

44. $x^{\frac{15}{8}}yz^{\frac{7}{6}}$ by $x^{\frac{7}{8}}y^{\frac{1}{2}}z^{\frac{1}{6}}$.

45. $15x^{\frac{12}{5}}y^{2}z^{\frac{20}{3}}$ by $5x^{\frac{2}{5}}y^{\frac{1}{4}}z^{\frac{5}{6}}$.

46. $a^{-2}b^{-4}$ by $a^{-1}b^{-5}$.

47. $a^{-4}b^{3}c^{-7}$ by $a^{-6}b^{-2}c^{-5}$.

48. $x^{-3}y^{-1}z^{-4}$ by $xy^{-1}z^{4}$.

49. $12a^{6}b^{-2}c^{-5}d$ by $2a^{-1}b^{-3}d^{-1}$.

50. $a^{-7}b^{-1}c^{4}$ by $a^{7}b^{-1}c^{-4}d^{2}$.

51. $9a^{2}b^{-2}c^{-1}-15a^{2}c^{-4}$ by $3a^{-1}bc^{-1}$.

52. $24x^2y^3z+15x^{-2}y^2z^6-9xy^{-4}$ by $3x^{-2}y^3z$.

53. $35x^4yz^{-2}-21x^3y^{-3}z^4+7y^{-7}z^{10}$ by $-7x^2yz^{-2}$.

54. $15a^3b^3c^{-1}+30a^{-1}c^2d^3-10b^{-1}d^6$ by $-5a^{-1}b^2c^3d^5$.

55. $20x^7y^{-1}z^3-4y^2z^6-12x^{-3}y^{-1}$ by $4x^{-3}y^{-2}z^6$.

56. $28x^{\frac{5}{2}}yz^{-1}+16xy^{-2}z^{-\frac{5}{6}}-12x^{-\frac{1}{2}}y^{-5}z^{-\frac{2}{3}}$ by $4x^{-\frac{1}{2}}y^{-2}z^{-3}$.

57. $x^3+5x^{\frac{3}{2}}+6x$ by $x^{\frac{1}{2}}+3$.

58. $4x^{\frac{4}{3}}-3x^{\frac{3}{3}}y^{\frac{1}{2}}-y$ by $x^{\frac{2}{3}}-y^{\frac{1}{2}}$.

59. $8a^{\frac{4}{5}}-6a^{\frac{1}{2}}+a^{\frac{1}{5}}$ by $2a^{\frac{2}{5}}-a^{\frac{1}{10}}$.

60. $9a^{-2}+12a^{-1}+4$ by $3a^{-1}+2$.

61. $12a^{\frac{3}{2}}+35a+25a^{\frac{1}{2}}$ by $4a^{\frac{1}{2}}+5$.

62. $4x-10x^{\frac{4}{5}}-62x^{\frac{3}{5}}-30x^{\frac{2}{5}}$ by $2x^{\frac{1}{5}}+5$

63. $6a^2-4a-10+8a^{-1}$ by $3a^{-1}+4a^{-2}$.

64. $3-16a^{\frac{1}{2}}+28a-16a^{\frac{3}{2}}$ by $3-4a^{\frac{1}{2}}$.

65. $4ax^{\frac{7}{10}}-14a^{\frac{4}{5}}x^{\frac{3}{5}}-6a^{\frac{3}{5}}x^{\frac{1}{2}}+36a^{\frac{2}{5}}x^{\frac{2}{5}}$ by $2a^{\frac{1}{5}}+3x^{-\frac{1}{10}}$.

66. $x^3y^{-3}+x^{-3}y^3$ by $xy^{-1}+x^{-1}y$.

67. $x-y^{\frac{3}{2}}+x^{\frac{1}{2}}y-x^{\frac{3}{2}}y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

68. $a^{\frac{5}{2}}-a^2-4a^{\frac{3}{2}}+6a-2a^{\frac{1}{2}}$ by $a^{\frac{3}{2}}-4a^{\frac{1}{2}}+2$.

69. $24a^2-58a^{\frac{5}{3}}+5a^{\frac{4}{3}}-4a+3a^{\frac{2}{3}}+12a^{\frac{1}{3}}-32$ by $2a^{\frac{2}{3}}-3a^{\frac{1}{3}}-4$.

70. $x^2-4y+6xz^{\frac{1}{2}}+9z^{\frac{2}{3}}$ by $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.

71. $x^{\frac{5}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}-y^{\frac{5}{2}}$ by $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

72. $3a^{-1}-23a^{-2}-5a^{-3}+80a^{-4}+50a^{-5}$ by $a^{-1}-6a^{-2}-10a^{-3}$.

73. $a^{\frac{4}{3}}b^{-\frac{3}{2}}-49a^{\frac{3}{2}}b^{-\frac{1}{2}}+56a^{\frac{1}{3}}-16b^{\frac{1}{2}}$ by $a^{\frac{2}{3}}-7a^{\frac{1}{3}}b^{\frac{1}{2}}+4b$.

74. $x^{\frac{5}{2}}-\frac{1}{2}x^{\frac{1}{2}}$ by $x^2+\frac{1}{2}$ (to 4 terms in the quotient).

75. $x^{3p}-y^{3q}$ by x^p-y^q.

76. $x^{\frac{3n}{2}}-x^{-\frac{3n}{2}}$ by $x^{\frac{n}{2}}-x^{-\frac{n}{2}}$.

Find the square root of

77. $16x^{\frac{3}{2}}-24x+9x^{\frac{1}{2}}$.

78. $49a^{-2}-56a^{-\frac{3}{2}}b^{\frac{1}{2}}+16a^{-1}b$.

79. $x^{\frac{3}{3}}-2x^{\frac{1}{2}}+3x^{\frac{1}{3}}-2x^{\frac{1}{6}}+1$.

80. $25x^{\frac{4}{3}}-30a^{\frac{1}{4}}x+49a^{\frac{1}{2}}x^{\frac{2}{3}}-24a^{\frac{3}{4}}x^{\frac{1}{3}}+16a$.

81. $4x^{\frac{8}{5}}-4x^2y^{-1}-7x^{\frac{4}{5}}y^{-2}+4x^{\frac{2}{5}}y^{-3}+4y^{-4}$.

82. $a^4y^{-\frac{4}{3}}-4a^{\frac{5}{2}}y^{-\frac{1}{3}}+6ay^{\frac{2}{3}}-4a^{\frac{1}{2}}y^{\frac{5}{3}}+a^{-2}y^{\frac{8}{3}}$.

Find the fourth root of

83. $16a^2-160ax^{\frac{1}{2}}+600x-1000a^{-1}x^{\frac{3}{2}}+625a^{-2}x^2$.

84. $81x^{\frac{2}{5}}y-108y^{\frac{1}{2}}+54x^{-\frac{1}{5}}-12x^{-\frac{4}{5}}y^{-\frac{1}{2}}+x^{-\frac{6}{5}}y^{-1}$.

CHAPTER III.

Products of Factors.

14. When an expression is composed of brackets unconnected by any sign, its value can always be found by multiplying together the quantities in the various brackets; but in many cases the student may learn after a little experience to determine the result by inspection.

15. I. $(a+b)(a-b)=a^2-b^2$;

for by actual multiplication

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

we see therefore that if the terms in the two brackets are the same but connected by a different sign, their product is the difference of the squares of the terms: this is expressed by the formula *the product of the sum and difference of two quantities is the difference of their squares.*

Exs. $(a+3b)(a-3b)=a^2-9b^2$.

$(5a^2+4x^3)(5a^2-4x^3)=25a^4-16x^6$.

$(2a-b)(2a+b)(4a^2+b^2)=(4a^2-b^2)(4a^2+b^2)$

$=16a^4-b^4$.

16. II. $(a\pm b)^2=a^2\pm 2ab+b^2$;

for by actual multiplication

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ \quad ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

hence the square of any binomial is found to be the *sum of the squares of its two terms, together with twice their product.*

Exs. $(3a+2b)^2=9a^2+12ab+4b^2.$

$(5x-2y^2)^2=25x^2-20xy^2+4y^4.$

$(a+3b)^2(a-3b)^2=(a^2-9b^2)^2=a^4-18a^2b^2+81b^4.$

17. III. $(a\pm b)(a^2\mp ab+b^2)=a^3\pm b^3.$

The student can verify this by actual multiplication as in the other cases: he must observe that when the result is the *sum* of the cubes (a^3+b^3), the smaller factor has also the + sign, and the larger factor has − before its second term; when the result is the *difference* of the cubes (a^3-b^3), the smaller factor has also the − sign, and the larger factor contains only positive signs.

Exs. $(a+3b)(a^2-3ab+9b^2)=a^3+27b^3.$

$(2x^2-y^2)(4x^4+2x^2y^2+y^4)=8x^6-y^6.$

$5(a^2-b^2)(a^4+a^2b^2+b^4)=5a^6-5b^6.$

18. IV. $(ax+b)(cx+d)=acx^2+x(ad+bc)+bd.$

This may at once be verified by multiplication, and with its help we can by inspection determine the product of any two binomials. The student will observe that in the result, the first term is the product of the first terms of the factors; the second term is found by multiplying the

first term of each factor by the last term of the other, and adding their products; the third term is the product of the last terms of the factors.

Exs. $(3x+4)(2x+7)=6x^2+29x+28$.

In this example, the first term in the result is the product of $3x$ and $2x$; to obtain the second term, multiply $3x$ by $7\,(=21x)$ and $2x$ by $4\,(=8x)$, and add the results $(=29x)$; the third term is the product of 7 and 4.

$$(4x^2+3y)(x^2+2y)=4x^4+(8+3)x^2y+6y^2$$
$$=4x^4+11x^2y+6y^2.$$

$$(2xy+z^2)(5xy+4z^2)=10x^2y^2+13xyz^2+4z^4.$$

19. When the signs in the brackets are not both positive, a corresponding change is involved in the result; the cases which may occur besides that we have considered are—(1) if both the signs are negative, the last term of the product will be positive, but the second term will be negative; (2) if one is positive and the other negative, the last term of the result will always be negative, but the sign of the second term will depend upon the greater of the two products which compose it being positive or negative.

Exs. $(x-7)(x-2)=x^2-9x+14$.

$(x-4)(x-15)=x^2-19x+60$.

$(3x+5)(2x-7)=6x^2-11x-35$.

$(4x+3)(5x-4)=20x^2-x-12$.

$(2x+7)(3x-4)=6x^2+13x-28$.

$(5x-3)(7x+6)=35x^2+9x-18$.

20. We shall here collect the results of this chapter for the sake of reference :

$(a+b)(a-b)=a^2-b^2$ ..... (1),

$(a\pm b)^2=a^2\pm 2ab+b^2$ ..... (2),

$(a\pm b)(a^2\mp ab+b^2)=a^3\pm b^3$ ..... (3),

$(ax+b)(cx+d)=acx^2+x(ad+bc)+bd$ ..... (4).

Exercise III.

Simplify the following expressions:

1. $(x+y)(x-y)$.
2. $(x+2y)(x-2y)$.
3. $(2x^2+a)(2x^2-a)$.
4. $(3x^2-4y^2)(3x^2+4y^2)$.
5. $(5xy-z^2)(5xy+z^2)$.
6. $(3a^3-bc^2)(3a^3+bc^2)$.
7. $(x+y)(x-y)(x^2+y^2)$.
8. $(a+2b)(a-2b)(a^2+4b^2)$.
9. $(a^2-c^2)(a^2+c^2)(a^4+c^4)$.
10. $(3a+2)(3a-2)(9a^2+4)$.
11. $2(x^2-3a)(x^2+3a)(x^4+9a^2)$.
12. $3(4x^2+25y^4)(2x+5y^2)(2x-5y^2)$.
13. $(x+y)^2$.
14. $(k+l)^2$.
15. $(3a+b)^2$.
16. $(4a-3b)^2$.
17. $(6a+7b)^2$.
18. $(9x^2-2y)^2$.
19. $(12ab^2+5c)^2$.
20. $(4x^2yz+7a^2)^2$.
21. $(3x-7y^2)^2$.
22. $(4x^2y-7z^2)^2$.
23. $5(2k^2-3l)^2$.
24. $12(a^2b-3c^3)^2$.
25. $(3x+y)^2(3x-y)^2$.
26. $(2a+7x)^2(2a-7x)^2$.
27. $3(x-y)^2(x+y)^2(x^2+y^2)^2$.
28. $2(2a-c)^2(2a+c)^2(4a^2+c^2)^2$.
29. $2(x^2-3z^2)^2-(x^2-6z^2)^2$.
30. $4(7a+x)(7a-x)+(a-2x)^2$.
31. $(x+y)(x^2-xy+y^2)$.
32. $(k-l)(k^2+kl+l^2)$.

33. $(x^2+y^2)(x^4-x^2y^2+y^4)$. 34. $(x^3-z^3)(x^4+x^3z^3+z^4)$.

35. $(x+2)(x^2-2x+4)$. 36. $(2x+3)(4x^2-6x+9)$.

37. $(4k+5l)(16k^2-20kl+25l^2)$.

38. $(2a^2-bc)(4a^4+2a^2bc+b^2c^2)$.

39. $(a^3-8c^3)(a^3+8c^3)-(a^3-4c^2)(a^4+4a^2c^2+16c^4)$.

40. $4(3x^2+y^2)(9x^4-3x^2y^2+y^4)-(5x^3+2y^3)^2$.

41. $(x+3)(x+2)$. 42. $(x+7y)(x+5y)$.

43. $(a+5x)(a+10x)$. 44. $(3a+2b)(4a+7b)$.

45. $(x-2y)(x-3y)$. 46. $(x-8)(x+2)$.

47. $(4x-9)(3x+2)$. 48. $(2x^2-5y)(5x^2-3y)$.

49. $(3a+2x)(a-5x)$. 50. $(7a+5x)(2a-3x)$.

51. $(11a-12x)(a+4x)$. 52. $(2a-7x)(11a+5x)$.

53. $2(3x^2-yz)(2x^2+3yz)$. 54. $5(x^2-3y)(3x^2-7y)$.

55. $7(ab-2c^2)(3ab+4c^2)$. 56. $4(3ax+y^2)(7ax-2y^2)$.

57. $(a+b)(x+y)+(a-b)(x-y)$.

58. $a(ax-4by)-(a+2b)^2(x-y)$.

59. $x(x+1)(x+2)+x(x-1)(x+2)-2x(x-1)(x+1)$.

60. $3(a-2x)^2+2(a-2x)(a+2x)+(3x-a)(3x+a)-(3x-a)^2$.

CHAPTER IV.

Resolution into Factors.

21. The Resolution of expressions into their elementary factors is the converse of the work of the last chapter; the formulæ given in Art. 20 will apply here also: e.g. from knowing that $(a+b)(a-b)=a^2-b^2$ we see at once that a^2-b^2 has for its elementary factors $a+b$ and $a-b$.

22. It will be observed that the *difference* of two squares can always be resolved into real factors, but that the *sum* of two squares cannot be so resolved.

Exs. $25x^2 - 9y^2 = (5x + 3y)(5x - 3y)$.

$4a^2 + 49b^2$ has no real factors.

$a^4 - 16b^4 = (a^2 + 4b^2)(a^2 - 4b^2)$

$= (a^2 + 4b^2)(a^2 + 2b)(a^2 - 2b)$.

23. When an expression is proposed for resolution, the student must first examine it to see if there be any figure or letter contained in all its terms; if there be, such figure or letter must first be taken out of the expression and written as a factor.

Exs. $3x^4 - 27y^2 = 3(x^4 - 9y^2) = 3(x^2 + 3y)(x^2 - 3y)$.

$8a^3x^2y - 50ay^5 = 2ay(4a^2x^2 - 25y^4)$

$= 2ay(2ax + 5y^2)(2ax - 5y^2)$.

24. An expression of three terms is sometimes a perfect square; this can readily be determined by observing if the first and last terms are squares, *and* if the middle term is twice the product of the roots of the first and last.

Exs. $4x^2 + 12xy + 9y^2 = (2x + 3y)^2$.

$64a^4 - 16a^2b + b^2 = (8a^2 - b)^2$.

$32ax^2 - 112a^3x + 98a^5 = 2a(16x^2 - 56a^2x + 49a^4)$

$= 2a(4x - 7a^2)^2$.

25. A binomial expression consisting of the sum or difference of two cubes can always be resolved into factors by the formula $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$; see the remarks in Art. 17.

Exs. $x^6 + 8y^3 = (x^2 + 2y)(x^4 - 2x^2y + 4y^2)$.

$64a^3 - 27b^3 = (4a - 3b)(16a^2 + 12ab + 9b^2)$.

$24ax^3 + 81a^4 = 3a(8x^3 + 27a^3)$

$= 3a(2x + 3a)(4x^2 - 6ax + 9a^2)$.

26. To learn how to resolve a trinomial expression which is not a perfect square, the student should first practise himself in those whose first terms have unity as their coefficient, e.g. $x^2+7x+12$: he will observe that for such quantities, the last terms of the factors must be such that they will make the last term of the given expression by *multiplication*, and the second term by *addition or subtraction*: thus in the above expression, the last term (12) might be formed of 12 and 1, or 6 and 2, or 4 and 3; but the first of these pairs would give 13 for the second term, the second would give 8; the last alone gives 7; hence 4 and 3 are the only figures which satisfy both conditions, and we have $x^2+7x+12=(x+3)(x+4)$.

Similarly $x^2-7x+12=(x-3)(x-4)$.

27. If the last term of the given expression be negative, we know that of the factors which compose it one must be positive and the other negative; hence the figures must be chosen so that their *difference* will give the coefficient of the middle term; thus in the expression

$$x^2-5x-24,$$

we see at once that the figures which would form the last term are 24 and 1, 12 and 2, 8 and 3, or 6 and 4, and of the pair selected one will be positive and the other negative. Now of these sets 8 and 3 alone give the coefficient 5, and we attach the $-$ sign to 8 because the difference is negative, which shews that the greater of the two figures is also negative: we have then

$$x^2-5x-24=(x-8)(x+3),$$

and similarly $x^2+5x-24=(x+8)(x-3)$.

28. When the student has had ample practice in these more simple examples, he will be able gradually to attain facility in resolving those that are more difficult.

Exs. $3x^2+7x+2=(3x+1)(x+2)$.

$10x^2+21x+9=(5x+3)(2x+3)$.

$12x^2+8xy-15y^2=(6x-5y)(2x+3y)$.

$6x^3+40x^2y-14xy^2=2x(x+7y)(3x-y)$.

$3x^2-6xy-2x+4y=3x(x-2y)-2(x-2y)$

$=(3x-2)(x-2y)$.

In this example the factor $x-2y$ is multiplied by $3x$ and also by -2; it is therefore multiplied by $3x-2$, and this quantity is written as a factor in another bracket.

$4a^2-b^2+(2a-b)^2=(2a-b)(2a+b)+(2a-b)^2$

$=(2a-b)(2a+b+2a-b)$

$=4a(2a-b)$.

$(2a+b)^4-16(a-b)^4$

$=\{(2a+b)^2-4(a-b)^2\}\{(2a+b)^2+4(a-b)^2\}$

$=\{(2a+b)-2(a-b)\}\{(2a+b)+2(a-b)\}\{(2a+b)^2+4(a-b)^2\}$

$=3b(4a-b)(8a^2-4ab+5b^2)$.

Exercise IV.

Resolve the following expressions into their elementary factors:

1.	c^2-d^2.	2.	a^2-25.	3.	$4x^2-49$.
4.	$100a^2-b^2c^4$.	5.	$144x^4-y^2z^6$.	6.	$121a^2-64b^4c^2$.
7.	$16a^2x-x^3$.	8.	$32x^2-2y^2$.	9.	$12x^2y-75y^5$.
10.	$28a^2b^5-63bc^4$.			11.	a^4-b^4.

12. x^4-y^8. 13. x^4-81y^4. 14. $16a^8-x^4y^{12}$.

15. $81y^4-625z^8$. 16. $a^2x^4-a^6$. 17. x^5-xy^4.

18. $6x^7-96x^3y^8$. 19. $405a^2x^8-5a^6y^{12}$.

20. $48a^2b^4x^4-243a^6b^6$. 21. $x^2+2xy+y^2$.

22. $a^4+2a^2b^3+b^6$. 23. $x^6+2x^4y^2+x^2y^4$.

24. $a^2-2ab+b^2$. 25. $a^6-2a^3b^4+b^8$.

26. $a^2-6a^3b+9a^4b^2$. 27. $36x^2-60x^3y^2+25x^4y^4$.

28. $144a^2x^6+168a^4x^7+49a^6x^8$.

29. $121x^{10}y^2-132x^5y^4z^3+36y^6z^6$.

30. $81a^2x^6-36a^3x^3y^3z+4a^4y^4z^2$.

31. $75x^4-120x^2y^3+48y^4$. 32. $700ax^4-140a^3x^5+7a^5x^8$.

33. a^3+x^3. 34. a^6+x^6. 35. a^3-8b^3.

36. $27x^3+64$. 37. $8x^3-y^3$. 38. $125x^6+27y^3z^9$.

39. $64x^{11}-8x^2y^6$. 40. $24x^2y^3-75x^5y^6z^9$.

41. x^2+7x+6. 42. $x^2+13x+42$.

43. $x^2+12x+20$. 44. $x^2+18x+45$.

45. $x^2-12x+27$. 46. $x^2-13x+30$.

47. $x^2-39x+108$. 48. $x^2-57x+212$.

49. x^2-x-12. 50. $x^2-3x-70$.

51. $x^2-15x-16$. 52. $x^2+2x-35$.

53. $x^2+7x-60$. 54. $x^2+12x-13$.

55. $2x^2+7x+3$. 56. $3x^2+10x+8$.

57. $4x^2+16x+15$. 58. $12x^4+35x^2+25$.

59. $15x^6-17x^3+4$. 60. $20a^4-43a^2x+21x^2$.

61. $3x^2-8xy-11y^2$.
62. $9a^4b^2+33a^2b^2-12b^4$.
63. $35a^3+105a^2x^2+70ax^4$.
64. $105x^4-5x^3y^3-100x^2y^4$.
65. $ax+ay+bx+by$.
66. $ax-ay+bx-by$.
67. $ax^2-ay^2-bx^2+by^2$.
68. $3bcd-3bef+cd-ef$.
69. $4ckl+2ck^2-4lm-2km$.
70. $2a^2x^3-3b^2xy^2+2a^2x^2y-3b^2y^3$.
71. $a^2bx^2+ab^2xy+acdxy+bcdy^2-aefxz-befyz$.
72. a^2-b^2+a-b.
73. $3(x^2-y^2)+6x+6y$.
74. $x^4-4y^4+x^2+2y^2$.
75. $x^2-y^2+(x-y)^2$.
76. $(a+b)^2-(a-b)^2$.
77. $(2x+y)^2-4(x+y)^2$.
78. $(x+3y)^4-(x-3y)^4$.
79. $(a+b)^3-(a-b)^3$.
80. $4a^2b^2-(a^2+b^2-c^2)^2$.
81. $(a^2+4ab+3b^2)^2-(a^2-4ab+3b^2)^2$.
82. $(3x^2-4x^2y+2xy^2)^2-4x^2(x+y)^4$.
83. $(a+b)^2-11c(a+b)+30c^2$.
84. $(a+b)^2(a-b)-(a-b)^2(a+b)$.

CHAPTER V.

INVOLUTION.

29. WE have shewn in Chapter III. how to determine by inspection the square of any binomial expression; similar rules may be given for an expansion to any power; but after the fifth power the coefficients become so large that it is better to defer expansions to the sixth and higher powers until the student learns how to obtain

them by the rule known as the Binomial Theorem. We shall first state some formulæ for the lower powers which will apply to any binomial :—

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ ..... (i),

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ ..... (ii),

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ ..... (iii),

$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ ..... (iv),

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ ..... (v),

$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ ..... (vi).

These results are all obtained by actual multiplication, but the following remarks may help the student to remember them:

α. When the terms of the binomial are connected by a negative sign, its expansion consists of terms alternately positive and negative.

β. The first and last terms always have for their *index* the power of the expansion; and each of the intermediate terms increases or decreases its indices by unity: thus in (vi) we find in the first and last terms a^5 and b^5 because the given quantity is $(a+b)^5$, and the intermediate terms have a^4b, a^3b^2,...

γ. The coefficients are the same in terms equidistant from each end; also for the first term the coefficient is unity, and for the second term it is always the same as the power of the expansion: thus in the value of $(a+b)^4$, the second term is $4a^3b$, and the second term from the end $4ab^3$.

Ex. 1. $(x+2y)^3 = x^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3$
$= x^3 + 6x^2y + 12xy^2 + 8y^3.$

This example depends upon (i); in it x takes the place of a, and $2y$ takes that of b; hence in finding the value, we first place $2y$ in a bracket with the index attached to b in the formula, and afterwards simplify each term.

Ex. 2. $(5x^3-2y)^3=(5x^3)^3-3(5x^3)^2(2y)+3(5x^3)(2y)^2-(2y)^3$
$=125x^9-3\times25x^6\times2y+3\times5x^3\times4y^2-8y^3$
$=125x^9-150x^6y+60x^3y^2-8y^3.$

In this case we use (ii), and for a write $5x^3$ in a bracket, and for b write $2y$ in another bracket.

Ex. 3. $(3k-2l^2)^5=(3k)^5-5(3k)^4(2l^2)+10(3k)^3(2l^2)^2$
$-10(3k)^2(2l^2)^3+5(3k)(2l^2)^4-(2l^2)^5$
$=243k^5-810k^4l^2+1080k^3l^4-720k^2l^6+240kl^8-32l^{10}.$

30. Hitherto the expressions which we have expanded have consisted of only two terms; there is however no difficulty in determining a rule for finding the square of an expression of three or any number of terms. By actual multiplication we find that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc,$$
$$=a^2+b^2+c^2+2a(b+c)+2bc.$$

So again $(a+b+c+d)^2=a^2+b^2+c^2+d^2$
$+2ab+2ac+2ad+2bc+2bd+2cd,$
$=a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2cd;$

and we should obtain a similar value however many terms we had in the expression.

From these results we deduce the following rule:—*The square of any multinomial is found by taking the square of each of its terms together with twice the product of each term and all the terms that follow it.* In practice

the student will find it easiest first to write down the square of the first term followed by twice the product of it and all the rest; then the square of the second term followed by twice the product of it and all the terms after it; and so on, till he comes to the square of the last term. He must afterwards remove the brackets and simplify the expression.

Ex. 4. $(x+2y-3z)^2 = x^2 + 2x(2y-3z)$
$+4y^2+4y(-3z)$
$+9z^2$
$= x^2+4xy-6xz+4y^2-12yz+9z^2$. Ans.

Ex. 5. $(x^3+4x^2-3x-2)^2 = x^6+2x^3(4x^2-3x-2)$
$+16x^4+8x^2(-3x-2)$
$+9x^2-6x(-2)$
$+4$
$= x^6+8x^5-6x^4-4x^3+16x^4-24x^3-16x^2$
$+9x^2+12x+4$
$= x^6+8x^5+10x^4-28x^3-7x^2+12x+4$. Ans.

31. When the cube or higher power of a trinomial or multinomial expression is required, two or more of the terms must be connected by a *vinculum* and treated as one term, and the expression can then be expanded as a binomial: afterwards the terms which have been temporarily formed into one term must also be expanded, and the whole expression will be ready for simplification.

Ex. 6. Expand $(x^2+2x+3)^3$.

We must treat $2x+3$ as one term; thus we have by Art. 29

$(x^2+2x+3)^3 = (x^2+\overline{2x+3})^3$
$= (x^2)^3+3(x^2)^2(2x+3)+3x^2(2x+3)^2+(2x+3)^3$
$= x^6+3x^4(2x+3)+3x^2(4x^2+12x+9)$
$+(8x^3+36x^2+54x+27)$
$= x^6+6x^5+21x^4+44x^3+63x^2+54x+27$. Ans.

Ex. 7. $(a^3-3a+4)^3=(a^3-\overline{3a+4})^3$

$=a^6-3a^4(3a+4)+3a^2(3a+4)^2-(3a+4)^3$

$=a^6-3a^4(3a+4)+3a^2(9a^2+24a+16)$

$-(27a^3+108a^2+144a+64)$

$=a^6-9a^5+15a^4+45a^3-60a^2-144a-64$. Ans.

32. If the product of some power of two or more expressions be required (the power being the same for all the expressions), first find the product of the simple expressions and then expand this result.

Ex. 8. $(a^2-x)^3(a^2+x)^3(a^4+x^2)^3=(a^4-x^2)^3(a^4+x^2)^3$

$=(a^8-x^4)^3=a^{24}-3a^{16}x^4+3a^8x^8-x^{12}$. Ans.

Exercise V.

Expand the following expressions:

1. $(x+y)^3$.
2. $(a^2+b)^3$.
3. $(k-l)^3$.
4. $(2x-y)^3$.
5. $(4x^2+yz)^3$.
6. $(5a-4b^2)^3$.
7. $(6x^2+5yz^3)^3$.
8. $(a+b)^4$.
9. $(c-d)^4$.
10. $(a+2d)^4$.
11. $(5a-c^2)^4$.
12. $(4x^3+yz^2)^4$.
13. $(2x^3-5yz^2)^4$.
14. $(3x-4y)^4$.
15. $(x+a)^5$.
16. $(a-c)^5$.
17. $(2x^3-y^2)^5$.
18. $(k+3l)^5$.
19. $(2a-3b^2)^5$.
20. $(a^3-2bc^2)^5$.
21. $(x+y+z)^2$.
22. $(x^2+y^2+z^2)^2$.
23. $(x^2+xy+y^2)^2$.
24. $(a^2-b^2+c^2)^2$.
25. $(a-3b-2c)^2$.
26. $(a^2-4ab-3b^2)^2$.
27. $(x^3-4x^2y+y^3)^2$.
28. $(p+q+r+s)^2$.
29. $(a+2b+3c+d)^2$.
30. $(a-2b-3c+d)^2$.
31. $(x^3-x^2+x+1)^2$.
32. $(1-2x+3x^2-x^3)^2$.
33. $(1-3x+4x^2-2x^3)^2$.
34. $(x^4+x^3+x^2-x-1)^2$.

35. $(a^4-2a^3b-4a^2b^2-2ab^3+b^4)^2$. 36. $(2x^4-3x^3-x+1)^2$.

37. $(a+b+c)^3$. 38. $(x-y+z)^3$. 39. $(x^2+x+1)^3$.

40. $(3x^3+2xy+y^2)^3$. 41. $(4x^4-2x^2+1)^3$.

42. $(1-x+2x^2)^3+(1+x+2x^2)^3$.

43. $(a^3+3a^2x+3ax^2+x^3)^3-(a^3-3a^2x+3ax^2-x^3)^3$.

44. $(a+4x)^3(a-4x)^3$. 45. $(a-x)^3(a+x)^3(a^2+x^2)^3$.

46. $(a-2x)^3(a^2+2ax+4x^2)^3$. 47. $(x+y^2)^4(x^3-xy^2+y^4)^4$.

48. $(x-4)^2(x+3)^2$. 49. $(2x-5)^3(x+3)^3$.

50. $(x^2-y^3)^3(x^3+y^3)^3(x^4+y^6)^3(x^8+y^{12})^3$.

CHAPTER VI.

HARDER EXAMPLES IN ELEMENTARY RULES.

33. IN the exercise at the end of this chapter the student will find examples in Addition, Subtraction, &c. in which the coefficients are fractional, or involve brackets. The following examples will suggest to him the methods by which he should proceed.

Ex. 1. Find the sum of

$$\frac{2}{3}x+\frac{y}{4}-\frac{z}{5},\ \frac{1}{6}x-\frac{3}{8}y-z,\ \text{and}\ 2x+\frac{1}{8}y-\frac{4z}{5}.$$

$\frac{2}{3}x+\frac{y}{4}-\frac{z}{5}$

$\frac{1}{6}x-\frac{3}{8}y-z$

$2x+\frac{1}{8}y-\frac{4z}{5}$

$\frac{17x}{6}-2z$. Ans.

The coefficients are added in the same way as when they are whole numbers; thus for x we have

$$\frac{2}{3}+\frac{1}{6}+2=\frac{17}{6};$$

so for y,

$$\frac{1}{4}-\frac{3}{8}+\frac{1}{8}=0;$$

and for z,

$$-\frac{1}{5}-1-\frac{4}{5}=-2.$$

Ex. 2. Divide

$$\frac{x^5}{6}-\frac{11x^4}{18}+\frac{7x^3}{12}-\frac{x}{2}+\frac{1}{4} \text{ by } \frac{x^2}{3}-x+\frac{1}{2},$$

$$\frac{x^2}{3}-x+\frac{1}{2}\Big)\frac{x^5}{6}-\frac{11x^4}{18}+\frac{7x^3}{12}-\frac{x}{2}+\frac{1}{4}\Big(\frac{x^3}{2}-\frac{x^2}{3}+\frac{1}{2}. \quad \text{Ans.}$$

$$\begin{array}{l}
\frac{x^5}{6}-\frac{x^4}{2}+\frac{x^3}{4} \\
\hline
\quad -\frac{x^4}{9}+\frac{x^3}{3}-\frac{x}{2} \\
\quad -\frac{x^4}{9}+\frac{x^3}{3}-\frac{x^2}{6} \\
\hline
\qquad\qquad \frac{x^2}{6}-\frac{x}{2}+\frac{1}{4} \\
\qquad\qquad \frac{x^2}{6}-\frac{x}{2}+\frac{1}{4} \\
\hline
\end{array}$$

34. When expressions which have the same factors in brackets are to be taken together, the brackets should be retained, as the work is thereby greatly simplified.

Ex. 3. From $4x(a^2-b^2)+3x^2(a+b)-x^3(a^2+b^2)$

subtract $4x(a^2-ab)-x^2(2a-3b)+2x^3ab$.

$$\begin{array}{l}
4x(a^2-b^2)+3x^2(a+b)-x^3(a^2+b^2) \\
4x(a^2-ab)-x^2(2a-3b)+2x^3ab \\
\hline
4x(ab-b^2)+x^2(5a)-x^3(a^2+b^2+2ab)
\end{array}$$

which may be written

$$4bx(a-b)+5ax^2-x^3(a+b)^2. \quad \text{Ans.}$$

Ex. 4. Divide $x^{10}+2ax^8+a^2x^6+(4a-1)x^4+a^2$
by $x^6+(a+1)x^4-(a-1)x^2+a$.

$x^6+(a+1)x^4-(a-1)x^2+a \;)\; x^{10}+2ax^8+a^2x^6+(4a-1)x^4+a^2 \;(\; x^4+(a-1)x^2+a$

$x^{10}+(a+1)x^8-(a-1)x^6+ax^4$ Ans.

$(a-1)x^8+(a^2+a-1)x^6+(3a-1)x^4+a^2$

$(a-1)x^8+(a^2-1)x^6-(a^2-2a+1)x^4+(a^2-a)x^2$

$ax^6+(a^2+a)x^4-(a^2-a)x^2+a^2$

$ax^6+(a^2+a)x^4-(a^2-a)x^2+a^2$

35. There are certain results in Division which can be obtained without any work by simple inspection; the student need not at present be troubled with the *proof* of the general rule, but he may with advantage have the rule put before him and be shewn how to apply it.

When n is an odd number

x^n+y^n is always divisible by $x+y$, and its quotient has its terms alternately + and −.

x^n-y^n is always divisible by $x-y$, and its quotient has all its terms +.

When n is an even number

x^n+y^n is *not* divisible either by $x+y$ or $x-y$.

x^n-y^n is divisible by both $x+y$ and $x-y$, the quotient following the same rule as to signs as when n is odd.

Thus, we can determine at once that x^7+y^7 is divisible by $x+y$, and its quotient has alternately + and − signs.

x^9-y^9 is divisible by $x-y$, and the quotient has only + signs.

x^6+y^6 is not divisible either by $x+y$ or $x-y$.

x^6-y^6 is divisible by both $x+y$ and $x-y$.

The application of the above results is rendered easier by observing that what is true for any small odd index is true also for a large one; and similarly for small and large even indices; hence if an expression with a large index is proposed we need only consider one much smaller, taking care however that both shall be odd or both even; thus we conclude that $x^{100}-y^{100}$ is divisible by both $x+y$ and $x-y$, because x^2-y^2 is so divisible; again $x^{135}+y^{135}$ is divisible by $x+y$ and not by $x-y$, because this is true of $x+y$.

Ex. 5. $(x^3+y^3)\div(x+y)=x^2-xy+y^2$.

Ex. 6. $(x^5-y^5)\div(x-y)=x^4+x^3y+x^2y^2+xy^3+y^4$.

Ex. 7. $(x^4-y^4)\div(x+y)=x^3-x^2y+xy^2-y^3$.

Ex. 8. $(x^4-y^4)\div(x-y)=x^3+x^2y+xy^2+y^3$.

Ex. 9. $(1-81x^4)\div(1-3x)=\{1-(3x)^4\}\div\{1-3x\}$
$=1+3x+9x^2+27x^3$.

Ex. 10. $(a^5+32b^{10})\div(a+2b^2)=\{a+(2b^2)^5\}\div\{a+2b^2\}$
$=1-2b^2+4b^4-8b^6+16b^8$.

36. In proving the identity of two given expressions, it is generally best to simplify the more complicated of the two and deduce the other from it; but if both are very long, it is sufficient to reduce them to the same simple quantity.

37. A very important method of simplifying an algebraical expression may be given here; it depends upon the following theorem which we shall first prove:—"Any algebraical expression which can be reduced to zero by supposing any letter or quantity in it to be zero, contains that letter or quantity as a factor."

Let S_a represent an algebraical expression containing a or powers of a, such that it is reduced to zero when a is put equal to 0; now suppose that when a is divided into S_a

it gives a quotient k and if possible a remainder l, then

$$S_a = ka + l;$$

this being true for all values of a, let $a=0$, then

$$S_0 = l;$$

but by the hypothesis

$$S_0 = 0, \quad \therefore \ l = 0,$$

i.e. there is no remainder;

$$\therefore \ S_a = ka,$$

and a is a factor of the given expression.

38. We will now apply this to the following example:—

Shew that $(a+b+c)^4 - (b+c)^4 - (c+a)^4 - (a+b)^4 + a^4 + b^4 + c^4 = 12abc(a+b+c)$.

In the given expression let $a=0$, and it becomes zero; hence a is a factor.

Similarly by making $b=0$, and $c=0$, we find that b and c are factors.

Next let $a+b+c=0$, and in this case also the expression becomes zero.

Hence we have $abc(a+b+c)$ as a factor in the value of the expression; and since this is of the same dimensions as the original expression, it is evident that there can be no other letters in the required value. It remains to see if there be any *numerical* coefficient; to determine this, let the expression $= Nabc(a+b+c)$ where N is an unknown number. Now since this is true for all values of a, b, and c, suppose that $a=b=c=1$, and we obtain $3^4 - 2^4 - 2^4 - 2^4 + 1 + 1 + 1 = 3N$;

$$\therefore \ 3N = 36, \quad \text{or } N = 12.$$

Hence the given expression $= 12abc(a+b+c)$.

Exercise VI.

Find the sum of

1. $\frac{1}{2}a+b+2c$, $\frac{5a}{6}+\frac{1}{3}b+\frac{1}{4}c$, and $\frac{2}{3}a+\frac{2}{3}b+\frac{3}{4}c$.

2. $x^3+\frac{1}{2}x^2-2$, $\frac{1}{3}x^3-\frac{1}{4}x^2+\frac{1}{5}x$, and $\frac{1}{2}x^3-\frac{1}{3}x^2-\frac{1}{2}x+\frac{3}{4}$.

3. $\frac{3a^3}{4}-ab^2+\frac{b^3}{5}$, $\frac{a^3}{4}-\frac{a^2b}{6}+\frac{b^3}{20}$, and $\frac{1}{2}(a^2b+ab^2)$.

4. $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{5}c$, $\frac{1}{4}a-\frac{1}{5}b-\frac{1}{3}c$, and $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c$.

5. $3x(a+b)-2y(c+d)$, $x(a+b)+10y(c+d)$, and $7(a+b)x$.

6. $(x-y)(a+b)+2x(a-b)$, $\frac{x(a+b)}{2}-2y(a-b)$, and $\frac{1}{2}(x+2y)(a+b)$.

7. $(a-2b)^2x-b^2y$, $(a-b)^2y+4abx$, and $a(2b-a)(x-y)$.

Subtract

8. $a+\frac{1}{3}b+\frac{1}{6}c+2d$ from $2a-\frac{1}{3}b+\frac{1}{6}c+\frac{1}{2}d$.

9. $\frac{x^3}{2}+\frac{2x^2}{3}-\frac{x}{5}+\frac{1}{9}$ from $x^3+\frac{5x^2}{3}+\frac{4x}{5}-\frac{2}{9}$.

10. $\frac{7a^4}{5}-\frac{a^2}{2}+2a$ from $2a^4-\frac{a^3}{3}+\frac{a^2}{2}-\frac{3a}{4}$.

11. $4a(x+y)-2b(x-y)$ from $5a(x+y)-3b(x-y)$.

12. $3x^2-\frac{a}{2}(x-y)+xy$ from $3x(x+y)-\frac{a}{4}(x-2y)+y(x+1)$.

13. $7(a^2-b^2)-\frac{c}{2}(a+b)+c^2$ from $10a(a+b)-ac-\frac{c^2}{3}$.

14. $2(a-b)x-3(a-2b)y$ from $2(x-y)a-3(x-2y)b$.

Find the product of

15. $\frac{a}{3}-\frac{b}{2}$ and $a+\frac{3b}{2}$.

16. $\frac{a^2}{2}+ab+\frac{b^2}{5}$ and $2a-5b$.

17. $\frac{x}{y}+1+\frac{y}{x}$ and $\frac{x}{y}-1+\frac{y}{x}$.

18. $\frac{3x^2}{y^2}-\frac{2x}{y}+1$ and $\frac{3x}{y}+2+\frac{y}{x}$.

19. $\frac{15b^2}{a^3}-\frac{7b^4}{a^5}+\frac{6b^6}{a^4}$ and $\frac{8b^2}{a^2}-\frac{3b^4}{a}$.

20. $(a+b)(x+y)-(a-b)(x-y)$ and $(a-b)(x+y)+(a+b)(x-y)$.

21. The product of two algebraical expressions is

$$4b^4-11ab^3-4a^2b^2+5a^3b-6a^4,$$

and one of them is $2(a^2+2b^2)-3ab$; find the other.

Divide

22. $4x^3+6-35x^5+58x^4-70x^3-23x$ by $6x^2-5x+2-7x^3$.

23. $x^4-\frac{19}{6}a^2x^2+\frac{a^3x}{3}+\frac{a^4}{6}$ by $x^2-2ax+\frac{a^2}{2}$.

24. $a^2+1+\frac{1}{a^2}$ by $a-1+\frac{1}{a}$.

25. $a^3-2+\frac{1}{a^3}$ by $\frac{1}{2a}-1+\frac{a}{2}$.

26. $(x-y)^2-z^2$ by $x-y+z$.

27. $(a^2+b^2)^2-8c^2d^2-4c^4-4d^4$ by $a^2+b^2-2(c^2+d^2)$.

28. $a^5+2ma^4+m^2a^3+(4m-1)a^2+m^2$ by $a^2+(m-1)a+m$.

29. $(3xy^3-2x^2y-2x^3+y^3)x$ by $y-x$.

30. $x^3-y^3-z^3+2yz+x+y-z$ by $y-z+x$.

31. $x^2(x+z)+y(x^2-z^2)-yz(x+y)$ by x^2-yz.

32. $x^4+(a+b)^2x^2+2ab(a+b)x-a^2b^2$ by $x^2-(a+b)x+ab$.

33. $a^2(x^3+x^2y)+2ab(x^2y+xy^2)+b^2(xy^2+y^3)$ by $ax+by$.

34. $(a^2-ax+x^2)(a^3-x^3)$ by $a^4+a^2x^2+x^4$.

35. $(a^2-2ab+b^2-c^2)(a+b+c)$ by $a-b-c$.

36. $(1-x^2)(1+x^2+x^4)(1+x^6+x^{12}+x^{18})$ by
$(1+x^6)(1-x^4+x^8)$.

37. $(a^2-2ab+3b^2)^3+(a^2+2ab+3b^2)^3$ by $2a^2+6b^2$.

38. $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$ by
$a+b+c$.

Write down the quotient of

39. x^5+y^5 divided by $x+y$.

40. x^3-8y^3 divided by $x-2y$.

41. x^7-y^7..... $x-y$.

42. $1+27x^3$ ..... $1+3x$.

43. x^4-81y^4..... $x-3y$.

44. x^6-y^6..... $x+y$.

45. $81x^4-16y^4$..... $3x+2y$.

46. x^8-y^8..... x^2+y^2.

47. x^9-125y^6 ..... x^3-5y^2.

48. $x^{10}-y^5$ ..... $3x^2-y$.

49. Which of the following expressions are divisible by $x+a$ or $x-a$?

$x^{350}-a^{350}$; $x^{351}+a^{351}$; $x^{452}+a^{452}$; $x^{553}-a^{553}$.

50. Which of the quantities

$$x^{127}-1,\ x^{126}-1,\ x^{125}+1,\ x^{124}+1$$

will divide by $x+1$ or $x-1$? Write down the last three terms in the quotients of those that will divide.

Prove the equality of the following expressions:—

51. $(ax+by)^2+(ay-bx)^2=(a^2+b^2)(x^2+y^2)$.

52. $a^2-m^2+b^2-n^2+2(mn-ab)$
$=(a+m-b-n)(a+n-b-m)$.

53. $a^2-ab+2(b^2-ab)+3(a^2-b^2)-4(a-b)^2$
$=5\{a^2-ab-(a-b)^2\}$.

54. $ab\{b-(a-c)\}-bc\{c-(a+b)\}+b(a-c)^2=2b^2(a+c)$.

55. $\{(a-b)^2+b^2\}\{(a+b)^2+b^2\}=a^4+4b^4$.

56. $(x+y+z)^2-(x^2+y^2+z^2)=x(y+z)+y(z+x)+z(x+y)$.

57. $\{(a+b)^2-c^2\}+\{(b+c)^2-a^2\}+\{(c+a)^2-b^2\}=(a+b+c)^2$.

58. $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2$
$=2(a-d)(a+b+c+d)$.

59. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$
$=(a+b+c-d)(a+b-c+d)(a-b+c+d)(b+c+d-a)$.

60. $(a-b)^3-(a+b)(a^2-b^2)+3ab(a-b)=(a+2b)(ab-b^2)$.

61. $(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)=abc$.

62. $4(a+b)(a-b)+6(a-b)^2-2(a+b)^2=8a(a-2b)$.

63. $(a-b)(b-c)(c-a)=ab(b-a)+bc(c-b)+ca(a-c)$.

64. $(b+c-a-d)^2(b-c)(a-d)+(c+a-b-d)^2(c-a)(b-d)$
$+(a+b-c-d)^2(a-b)(c-d)=0$.

65. $(a+b+c)(a^2+b^2+c^2+bc+ca+ab)-(b+c)^3-(c+a)^3$
$-(a+b)^3+(a+b+c)(a^2+b^2+c^2)=3abc.$

66. $(2a-b-c)^3+(2b-c-a)^3+(2c-a-b)^3$
$=3(2a-b-c)(2b-c-a)(2c-a-b).$

67. If $s=a+b+c$, prove that
$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$

68. If $3s=a+b+c$, prove that
$(s-a)^4+(s-b)^4+(s-c)^4=2\{(s-b)^2(s-c)^2+(s-c)^2(s-a)^2$
$+(s-a)^2(s-b)^2\}.$

CHAPTER VII.

LEAST COMMON MULTIPLE.

39. WE have previously given the rule for determining the L.C.M. of any two or more expressions by first finding their G.C.M. It may however be more readily determined by resolving the expressions into their factors. When the expressions consist of only one term, the result can be obtained by inspection; for example, suppose we require the L.C.M. of a^3b, a^2b^2c and a^4bc^3; it is clear that all the powers of a are contained in a^4; similarly those of b are contained in b^2; and those of c in c^3; hence $a^4b^2c^3$ will contain all the given expressions, i.e. it is their L.C.M. The same method will apply to any set of expressions with only one term; hence to find the L.C.M. we must *take the product of all the letters, each being written with the highest index it has in any of the expressions.* If there also be *numerical* coefficients, their L.C.M. is determined as in Arithmetic.

40. When the expressions contain more terms than one, each of them must be resolved into its elementary

factors, and the L.C.M. will be the product of these factors, each raised to the highest power that occurs in any of the expressions.

Ex. 1. Find the L.C.M. of

$$5x^3yz^2, \quad 10x^4y^2, \quad 15xy^2z^3.$$

The numerical coefficients are all contained in 30; also the highest power of x is x^4, which therefore contains all the other powers of x; similarly y^2 and z^3 contain all given powers of y and z: hence the L.C.M. is $30x^4y^2z^3$.

Ex. 2. Find the L.C.M. of

$$4a^2+4ab, \quad 2ab-2b^2, \quad 3a^2-3b^2.$$

The expressions may be written $4a(a+b)$, $2b(a-b)$, and $3(a^2-b^2)$. Now the factors outside the brackets are all contained in $12ab$; and a^2-b^2 is the product of $a+b$ and $a-b$: hence the L.C.M. is $12ab\,(a^2-b^2)$.

Ex. 3. Find the L.C.M. of

$$2x^2-2y^2, \quad x^3y+y^4, \quad \text{and } 3x^2+6xy+3y^2.$$

Resolving into factors, we have

$$2\,(x+y)\,(x-y), \quad y\,(x+y)\,(x^2-xy+y^2), \quad 3\,(x+y)^2.$$

It is evident that the last quantity $(x+y)^2$ contains the first bracket of each of the other expressions; hence they are all contained in

$$6y\,(x+y)^2(x-y)\,(x^2-xy+y^2)=6y\,(x^2-y^2)\,(x^3+y^3). \quad \text{Ans.}$$

Ex. 4. Find the L.C.M. of

$$2x^2+7xy+5y^2, \quad 4x^3+8x^2y+4xy^2, \quad 4xy^2+8xy^2-5y^3.$$

The expressions may be written

$$(2x+5y)\,(x+y), \quad 4x\,(x+y)^2, \quad y\,(2x+5y)\,(2x-y);$$

$\therefore$ the L.C.M. $=4xy\,(2x+5y)\,(x+y)^2(2x-y)$. Ans.

EXERCISE VII.

Find the L.C.M. of

1. a^2bc^3, ab^3c, a^3c^3.
2. $2a^4bx^2y$, $3a^2b^3y^3$, $6bx^6y$.
3. $12x^2yz^4$, $18xy^5z$, $20y^4z^3$.
4. $5kl^2x^3$, $4k^2xz^3$, $10k^3xyz$.
5. $6abc^3x^2$, $4a^3b^3x$, $5a^3cx^3$.
6. $15b^2c^3xz$, $9abx^2z^3$, $20a^2bcy^3$.
7. $2ab$, $3a(a+b)$.
8. $4abc$, $2ac(a-b)$, $3b(a-b)^2$.
9. $15x^2(x+y)$, $10xy(x+y)^3$, $24xy^2$.
10. $4x(x+y)^2$, $6x(x^2-y^2)$.
11. $3xy(x+z)$, $4xz(x+y)$, $6x^2y$.
12. $(a+b)^2$, $a-b$, a^2-b^2.
13. $4a^2(a^2+b^2)$, $8l^2(a^2-b^2)$, $3(a^4-b^4)$.
14. $8x^3(a+b)$, $3x(a+b)^2$, $12(a+b)^3$.
15. a^2+ab, $ab-b^2$, a^2-b^2.
16. k^3+kl^2, $5k^2l-5l^3$, $3k^2l+3kl^2$.
17. x^3+xy+y^2, $x-y$, x^3-y^3.
18. $3x(x^2-x+1)$, $2x^2(x+1)$, $4x^3+4$.
19. $a(a^3+8)$, $2a^2(a^3-2a+4)$, $(a+2)a^3$.
20. x^4+3x^3, $2x^4-6x^3+18x^2$, x^5+27x^2.
21. $8x^6y-27y^4$, $4x^4y^2+6x^2y^3+9y^4$, $2x^3-3xy$.
22. $a^2c+2abc^2+b^2c^3$, $3(a^3+b^3c^3)$, $2(a^4-a^2b^2c^2)$.
23. $x^5-2x^3yz^2+xy^2z^4$, $2(x^4y^3-y^4z^4)$, $x^4yz+x^2y^2z^3+y^3z^5$.
24. x^2+3xy, $xy-3y^2$, x^2-9y^2, x^2+9y^2.
25. x^2+7x+6, $x^2+11x+30$.
26. x^2-x-20, $x^2-9x+20$.

27. $3x^2+20x-7$, $2x^2+17x+21$, $2x^2-5x-12$.

28. $2x^2+7x-4$, $20x^3+2x-6$, $5x^3+23x+12$.

29. $6x^3-24xy^2$, $3x^2+5xy-2y^2$, $4x^2y+4xy^2-8y^3$.

30. $12a^3bc^3$, $16a^2(b^2+bc)^2$, $a^2b^3(ab+ac+cd+bd)$.

31. $ax+bx+ay+by$, $3(a^2-b^2)$, $(x+y)^2$.

32. $2x^2+x-6$, $ax+2a-x-2$, $3a(x^3+4x^2+4x)$.

CHAPTER VIII.

Addition and Subtraction of Fractions.

41. Fractions are added or subtracted in Algebra on the same principle as in Arithmetic; viz. first find the L. C. D. of the denominators, then express all the fractions with this denominator, and add or subtract the numerators.

Ex. 1. Find the value of

$$\frac{2x+y}{4}+\frac{3x-2y}{6}-\frac{x+3y}{8}.$$

The L. C. D. is 24; hence the expression becomes

$$\frac{12x+6y+12x-8y-3x-9y}{24}=\frac{21x-11y}{24}. \quad \text{Ans.}$$

The student will observe that the sign in the numerator of the last fraction is changed because of the $-$ sign which precedes it.

42. When the numerator or denominator of any fraction is itself fractional, the fraction must be simplified before it is combined with the others.

Ex. 2. Find the value of

$$\frac{\frac{4x}{3}-\frac{y}{6}}{5}-\frac{x-2y}{3\frac{1}{2}}+\frac{\frac{x}{4}-\frac{y}{5}}{10\frac{1}{2}}.$$

In the first fraction the L.C.D. of the terms in the numerator is 6; we therefore multiply both numerator and denominator by 6, and obtain $\frac{8x-y}{30}$.

In the second fraction the denominator has itself a denominator 2, and the fraction becomes $\frac{2x-4y}{7}$.

In the third fraction the L.C.D. of the fractions in the numerator and denominator is 20, and the fraction is equal to $\frac{5x-4y}{210}$.

Hence the expression becomes

$$\frac{8x-y}{30}-\frac{2x-4y}{7}+\frac{5x-4y}{210}$$

$$=\frac{56x-7y-60x+120y+5x-4y}{210}=\frac{x+109y}{210}. \quad \text{Ans.}$$

43. When the denominators of the fractions involve letters, the L.C.D. must be found by Chapter VII, the numerators raised to it, and added or subtracted as before.

Ex. 3. Add together

$$\frac{2a}{a+b}, \quad \frac{3a^2+b^2}{a^2-b^2}, \text{ and } \frac{b}{a-b}.$$

The L.C.D. is a^2-b^2; to form the numerator we divide a^2-b^2 by $a+b$, and multiply the quotient $a-b$ by $2a$; the second numerator is not changed as its denominator is a^2-b^2; for the third fraction we divide a^2-b^2 by $a-b$, and multiply the quotient $a+b$ by b. Thus the expression becomes

$$\frac{2a^2-2ab+3a^2+b^2+ab+b^2}{a^2-b^2}=\frac{5a^2-ab+2b^2}{a^2-b^2}. \quad \text{Ans.}$$

Ex. 4. Find the value of

$$\frac{2x}{3x-y} - \frac{x+3y}{3x+4y}.$$

The expression

$$= \frac{6x^2+8xy-3x^2-8xy+3y^2}{(3x-y)(3x+4y)} = \frac{3x^2+3y^2}{(3x-y)(3x+4y)}.$$

In an example like this, where a negative sign connects the fractions, the student must be careful to multiply the second numerator without reference to the sign before it, and *afterwards change all the signs of the product.*

44. If one or more of the denominators contain factors, they must be written with a bracket before the addition is begun, as in the following example.

Ex. 5. Find the value of

$$\frac{2}{3ax+3a^2} - \frac{3x}{2x^2+2a^2} + \frac{2a}{x^3-x^2a+xa^2}.$$

The expression may be written

$$\frac{2}{3a(x+a)} - \frac{3x}{2(x^2+a^2)} + \frac{2a}{x(x^2-xa+a^2)}$$

$$= \frac{4x^3-4x^2a+4xa^2-9x^2a+12xa^2+12a^3}{6xa(x^3+a^2)}$$

$$= \frac{4x^3-13x^2a+16xa^2+12a^3}{6xa(x^3+a^2)}.$$

45. Ex. 6. Add together

$$\frac{2}{(a+b)(a+c)}, \quad \frac{1}{(a+c)(b+c)} \text{ and } \frac{3}{(a+b)(b+c)}.$$

The required sum

$$= \frac{2b+2c+a+b+3a+3c}{(a+b)(a+c)(b+c)}$$

$$= \frac{4a+3b+5c}{(a+b)(a+c)(b+c)}. \quad \text{Ans.}$$

Ex. 7. Find the value of

$$\frac{2a}{(a-b)(a-c)} + \frac{3c}{(c-a)(b-c)} - \frac{2b}{(c-b)(b-a)}.$$

The letters in the factors must in this case be transposed; thus $(c-a)$ is the same as $-(a-c)$, and therefore the second fraction may be written $-\frac{3c}{(a-c)(b-c)}$; in the third fraction both factors are to be transposed, and as they *both* thereby become negative, the result is not changed; the expression may then be written

$$\frac{2a}{(a-b)(a-c)} - \frac{3c}{(a-c)(b-c)} - \frac{2b}{(b-c)(a-b)}$$

$$= \frac{2ab-2ac-3ac+3bc-2ab+2bc}{(a-b)(a-c)(b-c)}$$

$$= \frac{-5ac+5bc}{(a-b)(a-c)(b-c)} = \frac{-5c(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{-5c}{(a-c)(b-c)} = \frac{5c}{(c-a)(b-c)}. \quad \textbf{Ans.}$$

The student will observe that the result is here simplified by the cancelling of the factor $a-b$; this will suggest to him that he should always examine his result to see if any common factor is contained in the numerator and denominator: also, if the numerator in the result is a negative quantity, he should if possible change some factor of the denominator (as in this case $a-c$ is changed into $c-a$) so as to remove the $-$ sign from the numerator.

Ex. 8. Simplify

$$\frac{2x}{x^2-7x+12} - \frac{8}{x^2-x-12} + \frac{2}{x^2-9}.$$

This expression (when its denominators are resolved into factors) becomes

$$\frac{2x}{(x-4)(x-3)} - \frac{8}{(x-4)(x+3)} + \frac{2}{(x-3)(x+3)}$$
$$= \frac{2x^2+6x-8x+24+2x-8}{(x-4)(x-3)(x+3)}$$
$$= \frac{2x^2+16}{(x-4)(x^2-9)}. \quad \text{Ans.}$$

Exercise VIII.

Find the value of

1. $\frac{3x+y}{2} + \frac{2x+3y}{3}$.
2. $\frac{4x-2a}{5} + \frac{5x+7a}{15}$.
3. $\frac{a+2b+3c}{4} + \frac{2a-3b}{6}$.
4. $\frac{8a-3b+c}{10} + \frac{2a}{25} + \frac{a+c}{5}$.
5. $\frac{5x-2y}{4} - \frac{2x-3y}{6}$.
6. $\frac{x-2y+3z}{12} - \frac{2x-5z}{6} + \frac{3x-y}{8}$.
7. $\frac{2a-x}{2} - \frac{3a-4x}{3} - \frac{x+3y}{7}$.
8. $\frac{5x-a}{14} - \frac{x}{2} + \frac{1}{3} + \frac{x+3}{7}$.
9. $\frac{7x-2y-5}{10} - \frac{2(x-2)}{9} + \frac{3y-5}{15}$.
10. $\frac{2(x-1)}{3} - \frac{2x-3y}{4} - \frac{3x-5y-4}{6}$.
11. $\frac{\frac{a}{2}-4}{3} + \frac{\frac{2a}{3}-1}{6}$.
12. $\frac{\frac{3a}{5}-3}{2} + \frac{\frac{5a}{2}-2}{3} - \frac{a-1}{6}$.

13. $\dfrac{\frac{3a}{4}-\frac{b}{2}}{3}+\dfrac{a-\frac{2b}{3}}{2}-\dfrac{2a+\frac{4b}{5}}{4}$.

14. $\dfrac{8x-a}{9}-\dfrac{\frac{2x}{3}-a}{2}+\dfrac{\frac{3x}{4}-\frac{6a}{5}}{3}$.

15. $\dfrac{a+x-\frac{3y}{2}}{2}-\dfrac{a-\frac{3x}{4}}{3}+\dfrac{\frac{2x}{3}+3y}{4}$.

16. $\dfrac{3a-4x}{5}-\dfrac{\frac{5a}{2}+\frac{2x}{3}-\frac{y}{6}}{10}-\dfrac{\frac{a}{3}-\frac{x}{4}}{\frac{1}{2}}$.

17. $\dfrac{\frac{2a}{3}-b}{2}-\dfrac{a-\frac{b}{7}+\frac{c}{3}}{2}+\dfrac{\frac{2a}{7}+c}{3}$.

18. $\dfrac{5x^2}{6}-\dfrac{3x}{2}-\dfrac{\frac{x^2}{4}-x}{3}-\dfrac{x^2+\frac{3x}{2}-\frac{5}{3}}{6}$.

19. $\dfrac{5\left(x^2-\frac{x}{2}\right)}{11}+\dfrac{3\left(2x^2-\frac{1}{3}\right)}{2}-\dfrac{2\left(x-\frac{3}{4}\right)}{5\frac{1}{2}}$.

20. $\dfrac{a\left(7a-\frac{b}{3}\right)}{2}+\dfrac{2b\left(a+\frac{b}{4}\right)}{3}-\frac{1}{2}(ab+7a^2)$.

21. $\dfrac{1}{x+y}+\dfrac{1}{x-y}$.

22. $\dfrac{x}{x-y}-\dfrac{y}{x+y}$.

23. $\dfrac{a}{a+b}-\dfrac{b}{a-b}$.

24. $\dfrac{l}{k+l}+\dfrac{k}{k-l}$.

25. $\frac{x}{x+2a}+\frac{2a}{x-2a}$.

26. $\frac{a}{a-3b}-\frac{b}{a+3b}$.

27. $\frac{3}{x^2-y}-\frac{2}{x^2+y}$.

28. $\frac{a^2}{a^2-x^2}-\frac{x^2}{a^2+x^2}$.

29. $\frac{3k^2}{k^2-2l^2}-\frac{2l^2}{k^2+2l^2}$.

30. $\frac{7x^3}{x^3-y^3}+\frac{5y^3}{x^3+y^3}$.

31. $\frac{a}{a+x}+\frac{x}{a-x}-\frac{a^2}{a^2-x^2}$.

32. $\frac{x}{x-y}-\frac{y}{x+y}-\frac{x^2+y^2}{x^2-y^2}$.

33. $\frac{2a}{a-b}-\frac{3a}{a+b}+\frac{a^2}{a^2-b^2}$.

34. $\frac{3}{a-2}-\frac{2}{a+2}-\frac{10}{a^2-4}$.

35. $\frac{y^2}{x^2+y^2}-\frac{x^2}{x^2-y^2}+\frac{2(x^4+y^4)}{x^4-y^4}$.

36. $\frac{2}{x^2-3}-\frac{1}{x^2+3}-\frac{4}{x^4-9}$.

37. $\frac{2a}{x^3+a}+\frac{3x}{x^3-a}-\frac{x^4-2a^2}{x^6-a^3}$.

38. $\frac{x}{x-3a^2}-\frac{x^2}{x^2-9a^4}$.

39. $\frac{2}{x^2+2a}-\frac{x^2}{x^4-4a^2}$.

40. $\frac{2a^4}{a^4-9b^2}-\frac{a^2}{a^2+3b}$.

41. $\frac{x}{x-a}-\frac{a}{x+a}-\frac{x^2}{x^2-a^2}+\frac{a^2}{x^2+a^2}$.

42. $\frac{1}{x-2}-\frac{1}{x^2-4}-\frac{x+1}{x^2+4}$.

43. $\frac{a}{a-b}-\frac{b}{a+b}-\frac{b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$.

44. $\frac{a}{a^2-9}-\frac{1}{a+3}-\frac{3}{a^2+9}$.

45. $\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2-y^2}-\frac{x^4}{x^4-y^4}-\frac{y^4}{x^4+y^4}$.

46. $\frac{6}{2x-3}+\frac{4x}{2x+3}-\frac{2}{4x^2+9}$.

47. $\frac{3}{2(a+x)}-\frac{5}{2(a-x)}+\frac{2a+11x}{2(a^2-x^2)}$.

48. $\frac{x}{2(x+y)}+\frac{y}{3(x-y)}+\frac{xy}{6(x^2-y^2)}$.

49. $\frac{1}{5(3a-x^2)}-\frac{1}{2(3a+x^2)}+\frac{11a-7x^2}{10(9a^2-x^4)}$.

50. $\frac{x}{2(x-4a)}-\frac{2a}{x+4a}-\frac{x^2+16a^2}{3(x^2-16a^2)}$.

51. $\frac{x}{x+y}-\frac{x^2}{x^2-xy+y^2}+\frac{2x^2y}{x^3+y^3}$.

52. $\frac{2xy^2}{x^3+y^3}-\frac{y^2}{x^2-xy+y^2}+\frac{y}{x+y}$.

53. $\frac{x}{x-y}+\frac{x^2}{x^2+xy+y^2}-\frac{2x^3}{x^3-y^3}$.

54. $\frac{b}{a-b}+\frac{a^2}{a^2+ab+b^2}-\frac{a^3+b^3}{a^3-b^3}$.

55. $\frac{2x^2}{x^3+y^3}-\frac{x^4}{x^4-x^2y^2+y^4}-\frac{x^6-3x^4y^2}{x^6+y^6}$.

56. $\frac{a^6}{a^6+b^6}-\frac{a^4}{a^4-a^2b^2+b^4}+\frac{b^2}{a^2+b^2}$.

57. $\frac{b}{a^2+b}-\frac{a^4+b^2}{a^4-a^2b+b^2}+\frac{a^6}{a^6+b^3}$.

58. $\frac{x^2}{a-x^2}+\frac{x^4}{a^2+ax^2+x^4}-\frac{2ax^4}{a^3-x^6}$.

59. $\frac{2}{x+2} - \frac{x-3}{x^2-2x+4} - \frac{14}{x^3+8}$.

60. $\frac{18a}{8a^3-27} - \frac{1}{2a-3} + \frac{2a-3}{4a^2+6a+9}$.

61. $\frac{x}{x+2} + \frac{4}{x-4}$.

62. $\frac{a}{a-2b} - \frac{3a}{3a+b}$.

63. $\frac{2}{4x-y} - \frac{1}{2x+7y}$.

64. $\frac{4}{3x^2-2y^2} + \frac{2}{6x^2+y^2}$.

65. $\frac{3}{8x(2x+y)} + \frac{1}{12y(2x+y)}$.

66. $\frac{x}{3x(x-4a)} - \frac{2a}{6a(x-4a)}$.

67. $\frac{3}{x^2+xy} + \frac{2}{xy+y^2}$.

68. $\frac{3a}{ab+2b^2} - \frac{2b}{a^2+2ab}$.

69. $\frac{3}{4(x^2+3xy)} - \frac{1}{6(3y^2+xy)}$.

70. $\frac{2b}{a^2-ab} - \frac{3a}{b^2-ab}$.

71. $\frac{3}{x^2-xy} + \frac{2}{y^2-xy}$.

72. $\frac{a+3b}{a^2-3ab} - \frac{2a-b}{3b^2-ab}$.

73. $\frac{1}{x+1} + \frac{2}{x+2} - \frac{3}{x+3}$.

74. $\frac{x}{x+y} - \frac{y}{x-2y} + \frac{2y}{x+2y}$.

75. $\frac{a}{2a+b} + \frac{2a}{3a-b} - \frac{7a+b}{6a+b}$.

76. $\frac{2}{5a-b} - \frac{1}{2a-3b} + \frac{3}{a+b}$.

77. $\frac{x}{2(x+2y)} - \frac{y}{3(x+y)} + \frac{x}{6(x-y)}$.

78. $\frac{3}{a(a-x)} - \frac{1}{x(a+2x)} - \frac{4}{ax}$.

79. $\dfrac{10}{x(x+y)} - \dfrac{15}{2y(x-3y)} - \dfrac{8}{3xy}$.

80. $\dfrac{a}{a-c} - \dfrac{bc}{ab-ac} + \dfrac{c}{b-c}$.

81. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(a-c)(b-c)} + \dfrac{c}{(a-b)(b-c)}$.

82. $\dfrac{1}{(a-b)(b-c)} + \dfrac{2}{(a-c)(b-c)} - \dfrac{3a-2b}{(a-b)(a-c)(b-c)}$.

83. $\dfrac{x}{(x-y)(y-z)} - \dfrac{1}{x-z} - \dfrac{x-y}{(x-z)(y-z)}$.

84. $\dfrac{2k}{(k-l)(k-m)} - \dfrac{3m}{(k-m)(l-m)} + \dfrac{4l}{(k-l)(l-m)}$.

85. $\dfrac{3}{(b-c)(c-a)} - \dfrac{2}{(a-c)(a-b)} + \dfrac{1}{(b-a)(c-b)}$.

86. $\dfrac{2}{(y-z)(z-x)} - \dfrac{1}{(x-z)(x-y)} - \dfrac{3}{(y-x)(z-y)}$.

87. $\dfrac{2y}{(x-y)(z-y)} - \dfrac{3y}{(y-z)(z-x)} - \dfrac{5z}{(x-z)(x-y)}$.

88. $\dfrac{a}{(a-b)(a-c)} - \dfrac{2b}{(b-c)(b-a)} - \dfrac{3c}{(c-a)(b-c)}$.

89. $\dfrac{x+1}{x^2+3x+2} + \dfrac{x+2}{x^2+5x+6}$.

90. $\dfrac{3}{x^2+7x+12} - \dfrac{2}{x^2+2x-8}$.

91. $\dfrac{x+2}{x^2-x-12} - \dfrac{x}{x^2+6x+9}$.

92. $\dfrac{x}{x^2-7x-18} - \dfrac{1\frac{2}{3}}{x^2+7x+10}$.

93. $\dfrac{2x}{2x^2+x-15} - \dfrac{4x+2}{4x^2+4x-35}$.

94. $\dfrac{\frac{5}{2}}{3x^2-8x+5} + \dfrac{6x+1}{6x^2-13x+5}$.

95. $\dfrac{2}{x^2+3x+2} - \dfrac{1}{x^2+5x+4} - \dfrac{1}{x^2+6x+8}$.

96. $\dfrac{5x}{x^2-1} + \dfrac{\frac{3}{2}}{x^2-3x+2} - \dfrac{4x}{x^2-x-2}$.

97. $\dfrac{x}{4x^2-9} - \dfrac{1\frac{1}{2}}{2x^2-13x+15} + \dfrac{2}{2x^2+11x+12}$.

98. $\dfrac{\frac{2}{3}}{(x-4)(2x+1)} - \dfrac{\frac{x}{2}}{x^2+x-20} - \dfrac{3}{2x^2-9x-5}$.

99. $\dfrac{4}{x} + \dfrac{x}{4} + \dfrac{3}{x^2-3x} - \dfrac{7}{4x+12}$.

100. $\dfrac{4}{2x^2-5x} + \dfrac{3x}{x^2+3x-10} - \dfrac{7}{x-2} + \dfrac{4}{x+5}$.

CHAPTER IX.

Multiplication and Division of Fractions.

46. In Multiplication of Fractions a factor of any numerator may be cancelled with the same factor in its own or any other denominator; when such factors (if any) have all been removed, the required result is obtained by multiplying together all the numerators for a new numerator, and all the denominators for a new denominator.

Ex. 1. Simplify

$$\frac{3a^2b}{4c^3d} \times \frac{6acd^2}{7b^2c^3} \times \frac{14cd^3}{9a^2b^2}.$$

$$\frac{\cancel{3a^2b}}{\cancel{4c^3d}_{2c}} \times \frac{\cancel{6acd^2}^{3ad^2}}{7b^2c^3} \times \frac{\cancel{14cd^3}^{7d^2}}{\cancel{9a^2b^2}_{3b}} = \frac{ad^4}{b^3c^3}. \quad \text{Ans.}$$

We first cancel the first numerator into the last denominator; we then remove $2cd$ from the first denominator and last numerator; then we cancel the rest of the first denominator into the second numerator; and lastly, the numerical factors destroy each other.

47. When one of the given fractions is a divisor, it must be inverted, and then treated as in Multiplication.

Ex. 2. Simplify

$$\frac{4x^2}{3y^2} \times \frac{12xy}{7z^2} \div \frac{8x^3}{35yz^2}.$$

The expression

$$= \frac{4x^2}{3y^2} \times \frac{12xy}{7z^2} \times \frac{35yz^2}{8x^3}$$

$$= 10. \quad \text{Ans.}$$

48. If any numerator or denominator involve more terms than one, the student must be very careful not to remove any letter or figure unless it be a factor of the *whole* numerator or denominator; for example, if x^2+y^2 be a numerator, and x^2+xy a denominator, we cannot remove x from any of the terms, because there is one term in which it is not a factor; for the same reason y cannot be removed: but if we have x^2+xy as a numerator, and $xy+y^2$ as a denominator, although we cannot remove x or y separately, we can remove $x+y$, because it is a factor of the *whole* of both expressions.

49. Whenever it is possible, it is generally advisable to write the expressions with their factors in brackets.

Ex. 3. Find the value of

$$\frac{x^2-y^2}{2xy} \times \frac{x(x+2y)}{xy+y^2} \times \frac{3y^2}{x^2-xy}.$$

The expression

$$= \frac{\cancel{x^2-y^2}}{\cancel{2xy}} \times \frac{\cancel{x}(x+2y)}{\cancel{y}\,\cancel{(x+y)}} \times \frac{3\cancel{y^2}}{\cancel{x}\,\cancel{(x-y)}}$$

$$= \frac{3(x+2y)}{2x}. \quad \text{Ans.}$$

In this example it is easily seen that $x+y$ and $x-y$ in the denominators destroy the first numerator; similarly the simple factors are removed, and the result is readily obtained by taking the product of the remaining factors.

Ex. 4. Simplify

$$\frac{x^2+6xy+9y^2}{2x^2-5xy-7y^2} \times \frac{4x^3-22x^2y+28xy^2}{9x^2y+33xy^2+18y^3} \div \frac{2x^3-18xy^2}{3x^2y+2xy^2}.$$

The expression

$$= \frac{(x+3y)^2}{(2x-7y)(x+y)} \times \frac{2x(2x-7y)(x-2y)}{3y(3x+2y)(x+3y)} \times \frac{xy(3x+2y)}{2x(x^2-9y^2)}$$

$$= \frac{x(x-2y)}{3(x+y)(x-3y)}. \quad \text{Ans.}$$

Ex. 5. Find the value of

$$\left(1+\frac{2x}{a-x}\right) \times \frac{1}{x}\left(a-\frac{x^2}{a}\right) \div \left(1+\frac{a}{x}\right)^2.$$

In examples like this, the terms in the brackets must be simplified before the multiplication is commenced.

The first term $= \dfrac{a-x+2x}{a-x} = \dfrac{a+x}{a-x}$.

The second term $= \dfrac{1}{x} \times \dfrac{a^2-x^2}{a} = \dfrac{a^2-x^2}{ax}$.

The third term $= \frac{(x+a)^2}{x^2}$.

Hence the expression becomes

$$\frac{a+x}{a-x} \times \frac{a^2-x^2}{ax} \times \frac{x^2}{(a+x)^2}$$

$$= \frac{x}{a}. \quad \text{Ans.}$$

EXERCISE IX.

Find the value of

1. $\frac{a^2x}{b^2y} \times \frac{bcz}{axy} \times \frac{cy^2}{az^2}$.

2. $\frac{4ab^2}{5cd^2} \times \frac{15c^3d}{8b^3} \times \frac{6bcd}{a^3}$.

3. $\frac{12xyz}{25a^3} \times \frac{15a^2x}{16y^2z} \times \frac{10y^3}{9ax^2}$.

4. $\frac{5a^3}{7x^3} \times \frac{14xyz^2}{15a^2bc} \times \frac{b^3c^2}{2y^2z^3}$.

5. $\frac{a^2d^3}{5bc^4} \times \frac{15b^2cd^2}{28a^5} \div \frac{6bd^4}{7a^2c^3}$.

6. $\frac{2ax^2y}{3c^3z} \times \frac{15b^2yz}{4a^2cx} \div \frac{5ab^2y}{6c^4}$.

7. $\frac{x^2-y^2}{x^2} \times \frac{xy}{(x+y)^2}$.

8. $\frac{a-b}{a} \times \frac{b}{a+b} \times \frac{(a+b)^2}{ab}$.

9. $\frac{4x^2-y^2}{xy} \times \frac{x^2}{y(2x+y)} \times \frac{y^2}{(2x-y)^2}$.

10. $\frac{4a^2-9b^2}{ab} \times \frac{2a(a+b)}{b(2a+3b)} \times \frac{3b^2}{4(a^2-b^2)}$.

11. $\frac{7x(x+y)}{3y^2} \times \frac{6yz}{x^2} \times \frac{4x^2-36y^2}{x^2+2xy+y^2} \times \frac{y(x+y)}{35x(x+3y)}$.

12. $\frac{x^3+y^3}{x(x^2+y^2)} \times \frac{x^4+2x^2y^2+y^4}{y^2(x^2-xy+y^2)} \times \frac{2xy}{(x+y)^2}$.

13. $\frac{x^3-y^3}{2y(x^2+y^2)} \times \frac{x^4-2x^2y^2+y^4}{x^2(x^2+xy+y^2)} \times \frac{6xy}{x^2-y^2}$.

14. $\frac{4a^4-b^4}{8a^3+b^3} \times \frac{4a^2-2ab+b^2}{ab(a+b)} \times \frac{3a^2b}{2a^2+b^2}$.

15. $\dfrac{27x^6-8y^3}{2x^2(x^4-y^4)} \times \dfrac{3x^2+3y^2}{3x^2-2y} \times \dfrac{10xy}{9xy+9y^2}$.

16. $\dfrac{x^3+x^2y}{x^2y+y^3} \times \dfrac{3(x^4+2x^2y^2+y^4)}{4x^3y+4y^4} \times \dfrac{x^2y-xy^2+y^3}{3x^3+3xy^2}$.

17. $\dfrac{x^5-16xy^8}{7y^2(x^3+y^3)} \div \dfrac{x^5+4x^3y^4}{14(x^3y^2-x^2y^3+xy^4)}$.

18. $\dfrac{3a^6-81b^3c^9}{a^6-a^3b^6} \div \dfrac{9a^4+27a^2bc^3+81b^2c^6}{4a^4-4a^3b^3}$.

19. $\dfrac{a^2(x+3y)^2}{b^2(x^2+9y^2)} \times \dfrac{x^4-81y^4}{ab^2x+3ab^2y} \div \dfrac{2x^3y-18xy^3}{3b^4}$.

20. $\dfrac{x^6+64a^3}{(x^2+4a)^3} \times \dfrac{2x^6+16x^4a+32x^2a^2}{6x^4a^2+6x^2a^4} \div \dfrac{x^4-4x^2a+16a^2}{3a^2(x^2+a^2)}$.

21. $\dfrac{x^2+4x+3}{x^2+4x+4} \times \dfrac{x^2+7x+10}{x^2+8x+15}$.

22. $\dfrac{x^2+9x+20}{x^2-7x-18} \times \dfrac{x^2-8x-20}{x^2-5x-50}$.

23. $\dfrac{2x^2+7x+3}{3x^2+8x+4} \times \dfrac{9x^2+12x+4}{x^2+2x-3} \times \dfrac{3x^2-x-2}{2x^2-x-1}$.

24. $\dfrac{2x^2+13x-7}{2x^2+6x} \times \dfrac{2x^2+3x-9}{6x^2-7x+2} \times \dfrac{9x^2-6x}{4x^2-8x+3}$.

25. $\dfrac{x^3+2x^2y-63xy^2}{x^2y-6xy^2+8y^3} \times \dfrac{x^2y-2xy^2}{2x^2+22xy+36y^2} \div \dfrac{x^2-6xy-7y^2}{x^2+4xy+4y^2}$.

26. $\dfrac{2(x-2a)^3}{x(x^2+4a^2)} \times \dfrac{x^3+8a^3}{3a(x^2-4a^2)} \div \dfrac{x^4-4ax^3+4a^2x^2}{x^4+5a^2x^2+4a^4}$.

27. $\dfrac{12x(x^3+1)}{x^4+x^2} \times \dfrac{3x^4+6x^2+3}{(x^3-x+1)^2} \div \dfrac{24(x^3+x)^2}{x^4+1}$.

28. $\dfrac{x^4+9x^3-36x^2}{x^2a^2-10xa^2+9a^2} \times \dfrac{x^4-4x^2+3}{x^4-7x^3-18} \times \dfrac{a^2x^3+2a^2}{x^4+x^3}$.

29. $\dfrac{ax(x^3-4ax+4a^2)}{6(x^4+7a^2x^2+12a^4)} \times \dfrac{8(x^4+3a^2x^2-4a^4)}{a^2(x^2-4a^2)} \div \dfrac{x^4-x^2a^2}{x^2a^2+3a^4}$.

30. $\dfrac{54x^6-2y^6}{y^2(x^2+2y^2)^2} \times \dfrac{12(x^4+x^2y^2-2y^4)}{45x^4+15x^2y^2+5y^4} \times \dfrac{15x^2y^2+30y^4}{8(x^4-2x^3y+x^2y^2)}$.

31. $\left(1+\dfrac{x}{a}\right) \times \dfrac{a^2}{a^2-x^2} \times \left(\dfrac{a}{x}-1\right)$.

32. $\left(a+\dfrac{3x^2}{a}\right) \times \left(\dfrac{a^2}{3x^2}-1\right) \div \dfrac{a}{x^2}$.

33. $\left(1+\dfrac{2y}{x}+\dfrac{y^2}{x^2}\right) \times \dfrac{3xy}{x^2-y^2} \div \left(\dfrac{y}{x}+\dfrac{y^2}{x^2}\right)$.

34. $a\left(x-\dfrac{y^2}{x}\right) \times a^2\left(x^2+y^2+\dfrac{y^4}{x^2}\right) \div a^3\left(1-\dfrac{y}{x^6}\right)$.

35. $\dfrac{x^2-y^2}{3x^2} \times \left(1+\dfrac{2y}{x-y}\right)^2 \div \dfrac{(x+y)^3}{x^3-x^2y}$.

36. $\dfrac{a^3+b^3+3ab(a+b)}{2a(a-b)^2} \times b^2\left(a-\dfrac{b^2}{a}\right) \div \left(1+\dfrac{b}{a}\right)^4$.

37. $\dfrac{x^2+2xy+y^2-z^2}{x^2-2xy+y^2-z^2} \div \dfrac{x+y+z}{x-y+z}$.

38. $\dfrac{4a^2+b^2-c^2+4ab}{4a^2-b^2-c^2-2bc} \times \dfrac{2a-b-c}{2a+b+c}$.

39. $\dfrac{x^2-16y^2-9z^2+24yz}{x^2+16y^2-9z^2+8xy} \div \dfrac{x^2+16y^2-9z^2-8xy}{x^2-16y^2-9z^2-24yz}$.

40. $\left(1+\dfrac{8ab+7}{a^2-4ab+4b^2-16}\right) \times \dfrac{a^2-16-2b(a-4)}{a^2-9+2b(a-3)}$

$\div\left(1-\dfrac{1}{a-3}\right)$.

CHAPTER X.

Miscellaneous Fractions.

50. We shall now give various examples of the simplification of Fractions which do not fall under any of the rules in the two previous chapters.

51. When an expression consists of an integral quantity and a fraction, it may be reduced to an improper fraction by multiplying the integral part by the denominator and adding the numerator to this product; this will form the new numerator: the denominator remains unchanged. It will be observed that this is the same rule as in Arithmetic; thus to obtain $10\frac{2}{7}=\frac{72}{7}$ we multiply 10 by 7 and add 2, and so form the numerator 72; the denominator is 7 as before.

Ex. 1. Express as improper fractions

(i) $2+\frac{3b}{a}$. (ii) $x+2y-\frac{x^2-2y^2}{x-y}$.

In (i) to form the numerator we multiply 2 by a, and add $3b$; whence we obtain $2a+3b$: thus the result is $\frac{2a+3b}{a}$.

In (ii) we multiply $x+2y$ by $x-y$ $(=x^2+xy-2y^2)$; hence the expression becomes

$$\frac{x^2+xy-2y^2-x^2+2y^2}{x-y}=\frac{xy}{x-y}. \quad \text{Ans.}$$

52. The reverse process of expressing an improper fraction as a mixed quantity is performed by dividing the

numerator by the denominator; the quotient forms the integral part of the result, the remainder becomes the numerator of the fraction, and the denominator remains unaltered.

This also is the same process as in Arithmetic, the only difference being in the *form* of the result: the + sign which is understood in Arithmetic is expressed in Algebra; thus $11+\frac{2}{9}$ may be written $11\frac{2}{9}$ without fear of confusion, but $2a+\frac{a^2}{b}$ cannot be written $2a\frac{a^2}{b}$ because this form would lead to confusion between multiplication and addition.

Ex. 2. Express as mixed quantities

(i) $\frac{48x}{5}$. (ii) $\frac{x^2+3xy+y^2}{x+2y}$.

In (i) the quotient is $9x$, and the remainder $3x$; hence the result is $9x+\frac{3x}{5}$. Ans.

In (ii) the quotient is $x+y$ and the remainder $-y^2$; hence we obtain $x+y-\frac{y^2}{x+2y}$. Ans.

53. A *complex* fraction is one in which either the numerator or denominator (or both) contains fractional quantities. To find the value of a complex fraction, reduce both numerator and denominator to their simplest form, and multiply the numerator by the denominator inverted.

Ex. 3. Simplify

$$\frac{\frac{a+2x}{a-2x}-\frac{a}{a-4x}}{\frac{a-2x}{a+2x}-\frac{a}{a+4x}}.$$

The numerator

$$=\frac{a^2-2ax-8x^2-a^2+2ax}{(a-2x)(a-4x)}=\frac{-8x^2}{(a-2x)(a-4x)}.$$

The denominator

$$=\frac{a^2+2ax-8x^2-a^2-2ax}{(a+2x)(a+4x)}=\frac{-8x^2}{(a+2x)(a+4x)}.$$

Hence the fraction

$$=\frac{-8x^2}{(a-2x)(a-4x)}\times\frac{(a+2x)(a+4x)}{-8x^2}$$

$$=\frac{(a+2x)(a+4x)}{(a-2x)(a-4x)}$$

$$=\frac{a^2+6ax+8x^2}{a^2-6ax+8x^2}.$$ Ans.

54. To simplify a *continuous* fraction, the work must be commenced at the lowest part of the denominator; when that is reduced to its simplest form, substitute it in the fraction and combine it with the next part of the denominator, and so on, until the complete fraction is simplified.

Ex. 4. Reduce to its simplest form

$$1+\cfrac{1}{2-\cfrac{1}{2+\frac{1}{x}}}.$$

We begin with the lowest part of the denominator

$$\frac{1}{2+\frac{1}{x}}=\frac{1}{\frac{2x+1}{x}}=\frac{x}{2x+1}.$$

Substituting this in the given fraction, we have

$$1+\frac{1}{2-\frac{x}{2x+1}}=1+\frac{1}{\frac{4x+2-x}{2x+1}}$$

$$=1+\frac{2x+1}{3x+2}=\frac{5x+3}{3x+2}. \quad \text{Ans.}$$

EXERCISE X.

Express as improper fractions:

1. $1+\frac{x}{a}$.
2. $3+\frac{2a}{x-a}$.
3. $5-\frac{3x}{x+2y}$.
4. $2a-\frac{a^2}{a+x}$.
5. $a-\frac{a^2-b^2}{a+2b}$.
6. $3x-\frac{x(x+y)}{x-y}$.
7. $a+b-\frac{3b^2}{a-b}$.
8. $x+y-\frac{x^3}{x^2-xy+y^2}$.

Express as mixed numbers:

9. $\frac{12x}{7}$.
10. $\frac{25a}{11}$.
11. $\frac{38a^2}{15}$.
12. $\frac{217a}{50}$.
13. $\frac{a+2b}{a}$.
14. $\frac{x^2+xy+y^2}{x}$.
15. $\frac{6x^2-2y^2}{3x}$.
16. $\frac{4a^2-5ab}{2a}$.
17. $\frac{a^2+ab+b^2}{a+b}$.
18. $\frac{x^2-2xy+2y^2}{x-y}$.
19. $\frac{4a-3x}{4a+3x}$.
20. $\frac{2a-b-c}{2a+c}$.

Simplify the following complex fractions:

21. $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.
22. $\frac{\frac{1}{1+2x}+\frac{2x}{1-2x}}{\frac{1}{1-2x}-\frac{2x}{1+2x}}$.

23. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}$.

24. $\dfrac{\dfrac{1}{1+x}-\dfrac{x}{1-x}}{\dfrac{1}{1-x}-\dfrac{1-x}{x}}$.

25. $\dfrac{\left(1+\dfrac{2a}{x}\right)^2-\left(1-\dfrac{2a}{x}\right)^2}{\left(1+\dfrac{a}{x}\right)^2+\left(1-\dfrac{a}{x}\right)^2}$.

26. $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}}{\dfrac{x}{y}-\dfrac{y}{x}} \div \dfrac{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}{\left(\dfrac{1}{y}+\dfrac{1}{x}\right)^2}$.

27. $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}} \div \left\{\dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}\right\}^2$.

28. $\dfrac{\dfrac{3}{abc}}{\dfrac{1}{bc}+\dfrac{1}{ca}-\dfrac{1}{ab}}-\dfrac{3-a-b+c}{a+b-c}$.

29. $\dfrac{\dfrac{1}{a-x}-\dfrac{1}{a-y}+\dfrac{x}{(a-x)^2}-\dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}$.

30. $\dfrac{\left(a^5-\dfrac{1}{a}\right)}{\left(1+\dfrac{1}{a}\right)(a^3+a+1)(a^3-a+1)}$.

31. Divide $1+\dfrac{3ab}{a^2-ab+b^2}$ by $\dfrac{2a^2(a+b)}{a^3+b^3} \times \dfrac{a^2-b^2}{3ab}$.

32. Shew that $\dfrac{1+\dfrac{6x}{x^2-3x+2}}{1+\dfrac{2x}{x^2-x-2}}=1+\dfrac{4x}{(x-1)^2}$.

Find the value of

33. $1+\dfrac{1}{1+\dfrac{2}{x-2}}$.

34. $1-\dfrac{1}{1+\dfrac{3}{2x-3}}$.

35. $2+\dfrac{1}{1-\dfrac{a}{x+a}}$.

36. $3-\dfrac{2}{1-\dfrac{3x}{x+a}}$.

37. $2a-\dfrac{2a}{1-\dfrac{3a}{x+3a}}$.

38. $x-\dfrac{a}{2-\dfrac{4x}{2x+a}}$.

39. $2x-\dfrac{4x}{3-\dfrac{x}{x+2y}}$.

40. $3-\dfrac{1}{4-\dfrac{8}{2-\dfrac{a}{x}}}$.

41. $2+\dfrac{1}{3-\dfrac{1}{1-\dfrac{x}{y}}}$.

42. $1-\dfrac{1}{2-\dfrac{2}{1-\dfrac{2}{3y}}}$.

43. $3-\dfrac{1}{2-\dfrac{1}{2x-\dfrac{1}{1-\dfrac{1}{5x}}}}$.

44. $3-\dfrac{1}{2-\dfrac{1}{1-\dfrac{1}{2+\dfrac{1}{2+\dfrac{7x}{3-6x}}}}}$.

45. $2+\dfrac{1}{2-\dfrac{1}{1-\dfrac{1}{2-\dfrac{1}{1-\dfrac{1}{2+\dfrac{6x}{a-3x}}}}}}$.

Simplify

46. $\left(1-\dfrac{2xy}{x^2+y^2}\right)\left\{\dfrac{(x+y)^2+(x-y)^2}{(2x-y)^2-y^2}\right\}$.

47. $\left\{1-\dfrac{xy}{x^2+y^2}\right\}\left\{x^3+y^3+\dfrac{(x^3-y^3)^2}{x^3+y^3}\right\}$.

48. $\dfrac{2-x}{x^2-3x-28}+\dfrac{x}{x^2-6x-7}+\dfrac{-\frac{3}{8}}{x^2+5x+4}$.

49. $\dfrac{\frac{1}{3}}{2x^2-3x+1}-\dfrac{\frac{3x}{4}}{6x^2-x-1}+\dfrac{\frac{7}{8}}{1+2x-3x^2}$.

50. $\dfrac{1}{1-\left\{\dfrac{a-b-(ab-1)x}{ab-1-(a-b)x}\right\}^2}$

$$+\frac{\{(ab-1)-(a-b)x\}^2}{(1-ab)\{1+ab+(a+b)x\}-(a-b)\{a+b+(ab+1)x\}}.$$

Prove the following equalities:

51. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}=\dfrac{1}{abc}$.

52. $1-\left(\dfrac{x^2+y^2-z^2}{2xy}\right)^2$

$$=\frac{(x+y+z)(x+y-z)(x+z-y)(y+z-x)}{4x^2y^2}.$$

53. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}}\div\dfrac{1}{x+\dfrac{1}{y}}-\dfrac{1}{y(xyz+x+z)}=1.$

54. $\dfrac{(x+1)^2-4}{(x+1)^3-8}\div\dfrac{\frac{x}{2}+1\frac{1}{2}}{\frac{x^2}{7}+\frac{4}{7}x+1}=\dfrac{2}{7}$.

CHAPTER XI.

SIMPLE EQUATIONS.

55. THE Equations which we have hitherto considered have had only plain figures in their denominators: we shall now proceed to some in which the denominator contains the unknown quantity either with or without another term; in both cases the L.C.D. must be found as in Chapter VII. We can then remove the fractions from the equation by multiplying both sides of it by the L.C.D.

Ex. 1. Solve the equation

$$\frac{4}{3x}+\frac{5}{2x}=\frac{3}{2x}+\frac{7}{9}.$$

The L.C.D. is $18x$; then multiplying both sides by $18x$, we obtain

$$24+45=27+14x;$$

$$14x=42,$$

$$\therefore\ x=3.\quad \text{Ans.}$$

Ex. 2. Solve

$$\frac{3}{2(x+3)}-\frac{1}{x+4}=\frac{1}{x+3}.$$

The L.C.D. is $2(x+3)(x+4)$; hence the equation becomes

$$3(x+4)-2(x+3)=2(x+4),$$

$$3x+12-2x-6=2x+8;$$

$$\therefore\ x=-2.\quad \text{Ans.}$$

Ex. 3. Solve

$$\frac{2}{4x^2-1}-\frac{3}{1-2x}=\frac{11}{1+2x}.$$

We must first transpose the terms in the denominators so as to have the plain figure last in all; thus the equation becomes

$$\frac{2}{4x^2-1}+\frac{3}{2x-1}=\frac{11}{2x+1}.$$

The L. C. D. is $4x^2-1$, and we obtain

$$2+3(2x+1)=11(2x-1),$$

$$2+6x+3=22x-11;$$

$$16x=16;$$

$$\therefore x=1. \quad \text{Ans.}$$

The student should carefully observe the transposition performed in the above example; he should never begin the solution of an equation of this kind until he has arranged the terms with and without x in the same relative order in all the denominators.

Ex. 4. Solve

$$\frac{4}{3x-6}+\frac{1}{6-2x}=\frac{x+3}{3(42-35x+7x^2)}.$$

Rearranging the terms, we have

$$\frac{4}{3(x-2)}-\frac{1}{2(x-3)}=\frac{x+3}{21(x^2-5x+6)}.$$

The L. C. D. is $42(x-2)(x-3)$;

hence

$$56(x-3)-21(x-2)=2(x+3),$$

$$56x-168-21x+42=2x+6,$$

$$33x=132,$$

$$x=4.$$

56. **When the equation contains fractions in which the first term of the denominator will divide exactly into the first term of the numerator, the work is much abbreviated by expressing the fractions as mixed quantities.**

Ex. 5. Solve

$$\frac{5x-8}{x-2}-\frac{6x}{2x+3}=\frac{4x^2+9}{2x^2-x-6}.$$

The equation may be written

$$5+\frac{2}{x-2}-\left(3-\frac{9}{2x+3}\right)=2+\frac{2x+21}{2x^2-x-6}.$$

$$\frac{2}{x-2}+\frac{9}{2x+3}=\frac{2x+21}{2x^2-x-6},$$

$$4x+6+9x-18=2x+21,$$

$$11x=33,$$

$$x=3. \quad \text{Ans.}$$

Ex. 6. Solve $\frac{x-3}{x-5}+\frac{x+4}{x+2}=\frac{x-5}{x-7}+\frac{x+6}{x+4}$.

$$1+\frac{2}{x-5}+1+\frac{2}{x+2}=1+\frac{2}{x-7}+1+\frac{2}{x+4},$$

$$\frac{1}{x-5}+\frac{1}{x+2}=\frac{1}{x-7}+\frac{1}{x+4},$$

$$\frac{x+2+x-5}{(x-5)(x+2)}=\frac{x+4+x-7}{(x-7)(x+4)},$$

$$\frac{2x-3}{x^2-3x-10}=\frac{2x-3}{x^2-3x-28}.$$

Since $2x-3$ is the numerator of each fraction, it is evident that the equation will be satisfied by having $2x-3=0$ (for this will make both fractions vanish); hence

$$2x-3=0,$$

$$2x=3,$$

$$x=1\tfrac{1}{2}. \quad \text{Ans.}$$

57. When decimal fractions appear in the equation, the coefficients of the unknown letter may be placed in a bracket, and simplified by addition or subtraction: or, if the coefficients be complex quantities, the decimals may be at once expressed as vulgar fractions, and the equation solved in the usual manner.

Ex. 7. Solve

$$\cdot 25x + \cdot 125x + 10 = \cdot 2x + \cdot 75 + 1\cdot 1x.$$

The equation may be written

$$x(\cdot 25 + \cdot 125 - \cdot 2 - 1\cdot 1) = -10 + \cdot 75,$$

$$\cdot 925x = 9\cdot 25,$$

$$x = 10. \quad \text{Ans.}$$

Ex. 8. Solve

$$\cdot 75x - 2\cdot 25x = \frac{\cdot 5x - 2}{\cdot 2} - \frac{\cdot 2x - \cdot 5}{\cdot 01}.$$

$$\frac{75x}{100} - \frac{225x}{100} = \frac{\frac{5x}{10} - 2}{\frac{2}{10}} - \frac{\frac{2x}{10} - \frac{5}{10}}{\frac{1}{100}},$$

$$\frac{3x}{4} - \frac{9x}{4} = \frac{5x - 20}{2} - 10\,(2x - 5),$$

$$3x - 9x = 10x - 40 - 80x + 200,$$

$$64x = 160,$$

$$x = 2\tfrac{1}{2}. \quad \text{Ans.}$$

58. Some equations involving roots may be solved as simple equations: they frequently have in them two roots of expressions containing x; in such cases the roots must be placed on opposite sides of the equation, and both sides squared.

Ex. 9. Solve

$$\sqrt{4x+5}+2\sqrt{x-1}=9.$$

Transposing, we have

$$\sqrt{4x+5}=9-2\sqrt{x-1},$$

$$4x+5=81-36\sqrt{x-1}+4x-4,$$

$$36\sqrt{x-1}=72,$$

$$\sqrt{x-1}=2,$$

$$x-1=4,$$

$$x=5. \quad \text{Ans.}$$

EXERCISE XI.

Solve the following equations:

1. $\frac{4}{x}+3=\frac{11}{2x}-\frac{1}{2}$.

2. $\frac{6}{x}+7=\frac{36}{5x}+6\frac{2}{5}$.

3. $\frac{12}{x}-4=\frac{9}{4x}+5\frac{1}{4}$.

4. $\frac{9}{x}-\frac{4}{3x}=3\frac{1}{6}$.

5. $\frac{3}{2x}-\frac{5}{3x}=\frac{1}{6}$.

6. $\frac{8}{3x}-\frac{9}{2x}=\frac{11}{12}$.

7. $\frac{7}{x}-\frac{14}{3x}=\frac{1}{3}$.

8. $\frac{20}{3x}+6\frac{1}{3}=\frac{1}{x}+5\frac{1}{3}$.

9. $\frac{2}{x}-\frac{3}{2x}-\frac{1}{3x}=\frac{1}{3}$.

10. $\frac{1}{x}+\frac{1}{2x}=6+\frac{3}{4x}$.

11. $\frac{4}{x+1}+\frac{1}{3}=\frac{16}{3(x+1)}$.

12. $\frac{6}{x+2}+\frac{3}{4}=\frac{33}{4(x+2)}$.

13. $\frac{1}{5(x+3)}+\frac{2}{5}=\frac{1}{x+3}$.

14. $\frac{3}{2x+7}-\frac{2}{5}=\frac{9}{5(2x+7)}$.

15. $\frac{5}{x}=\frac{1}{x-4}$.

16. $\frac{1}{x}=\frac{2}{x+3}$.

17. $\frac{5}{x+3}=\frac{7}{x+5}$.

18. $\frac{3}{2x-1}=\frac{2}{x-1}$.

19. $\frac{1}{x-7}=\frac{15}{x}$.

20. $\frac{2}{x-1}=-\frac{6}{x+1}$.

21. $\frac{10}{x+20}+\frac{7}{3x-14}=0$.

22. $\frac{1}{5x+9}+\frac{3}{2x+7}=0$.

23. $\frac{3}{x+1}+\frac{1}{x-1}=\frac{6}{x^2-1}$.

24. $\frac{5}{x^2-4}=\frac{1}{x+2}+\frac{2}{x-2}$.

25. $\frac{12}{4x^2-9}-\frac{2}{2x+3}=\frac{1}{2x-3}$.

26. $\frac{2}{x+4}=\frac{11}{x^2-16}+\frac{1}{4-x}$.

27. $\frac{1}{9x+4}-\frac{2}{81x^2-16}+9x-4=0$.

28. $\frac{4}{6x-1}=\frac{11}{36x^2-1}+\frac{3}{6x+1}$.

29. $\frac{3}{x+3}-\frac{2}{4-x}=\frac{19}{(x+3)(x-4)}$.

30. $\frac{3}{x-4}-\frac{5}{2x-7}=\frac{5}{(x-4)(2x-7)}$.

31. $\frac{4}{2x+1}-\frac{7}{3x+2}=\frac{3}{(2x+1)(3x+2)}$.

32. $\frac{2}{4x-5}=\frac{9}{(4x-5)(2x-1)}+\frac{1}{1-2x}$.

33. $\frac{4}{2x-5}+\frac{3}{(7-3x)(5-2x)}=\frac{11}{3x-7}$.

34. $\frac{1}{x+4}=\frac{2}{2x-7}+\frac{15}{(7-2x)(4+x)}$.

35. $\frac{3x+13}{x+4}+\frac{4x-3}{x-2}=7.$

36. $\frac{7x-12}{x-2}+\frac{6x-17}{2x-5}=10.$

37. $\frac{3x+8}{x+3}-\frac{6x+13}{3x+7}=1.$

38. $\frac{2x+20}{x+7}-\frac{x+4}{x+1}=1.$

39. $\frac{3x+10}{x+2}-\frac{4x-20}{2x-5}=1.$

40. $\frac{8x-11}{x-2}-\frac{2x}{x+1}=2.$

41. $\frac{3x-9}{x-5}-\frac{2x-1}{2x-11}=2.$

42. $\frac{12x-4}{3x-2}-\frac{9-3x}{4-x}=1.$

43. $\frac{1}{3x}+\frac{3}{2x-4}=\frac{17}{6x}.$

44. $\frac{1}{x+3}+\frac{3}{10x}+\frac{1}{5x}=0.$

45. $\frac{1}{2x+5}-\frac{1}{3x}=\frac{11}{3(2x+5)}.$

46. $\frac{2}{3x+4}-\frac{1}{6x}=\frac{1}{2(3x+4)}.$

47. $\frac{3}{2x-2}-\frac{1}{x}=\frac{5}{6x-6}.$

48. $\frac{5}{2x-8}-\frac{1}{x-3}=\frac{7}{4x-16}.$

49. $\frac{5}{3x-6}-\frac{1}{2x+4}=\frac{6}{4x-8}.$

50. $\frac{2}{3x+9}-\frac{5}{4x+16}=\frac{1}{12x+48}.$

51. $\frac{1}{2x-10}+\frac{17}{x+5}=\frac{25}{2x+10}.$

52. $\frac{1}{2x+12}+\frac{2}{3x+6}+\frac{5}{6x+12}=0.$

53. $\frac{1}{2x+5}+\frac{45}{12x^2-75}=\frac{4}{4x-10}.$

54. $\frac{2}{x-4}-\frac{1}{2x-9}=\frac{7}{2x^2-17x+36}$.

55. $\frac{1}{2(3x+7)}-\frac{2}{3x^2+22x+35}+\frac{1}{2x+10}=0$.

56. $\frac{1}{4x^2+10x}+\frac{6x+16}{6x+15}=1-\frac{1}{6x}$.

57. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-7}{x-8}-\frac{x-8}{x-9}$.

58. $\frac{x}{x-1}-\frac{x-1}{x-2}=\frac{x-10}{x-11}-\frac{x-11}{x-12}$.

59. $\frac{x+4}{x+3}-\frac{x+3}{x+2}=\frac{x+6}{x+5}-\frac{x+5}{x+4}$.

60. $\frac{x+5}{x+4}+\frac{x+7}{x+6}=\frac{x+4}{x+3}+\frac{x+8}{x+7}$.

61. $\frac{1}{9}\left(2x-\frac{1}{5}\right)-\frac{3}{4}\left(\frac{x}{3}-\frac{2}{9}\right)=\frac{1}{5}\left(\frac{5x}{4}-1\right)+\frac{1}{15}$.

62. $\frac{3}{10}\left(x+\frac{1}{3}\right)+\frac{2}{5}\left(\frac{x}{6}+2\right)-\frac{13}{4}=\frac{7}{18}\left(\frac{9x}{14}-5\frac{1}{7}\right)$.

63. $\frac{x}{8}(4x+2\frac{2}{3})-\frac{2}{9}\left(\frac{3x}{4}-4\frac{1}{2}\right)=\frac{x}{6}(x-2)+\frac{x}{5}\left(\frac{5x}{3}+10\right)$.

64. $\frac{\frac{x}{6}-3}{2}+\frac{\frac{5x}{4}-2}{3}-\frac{\frac{2x}{3}-\frac{5}{6}}{2\frac{1}{2}}=\frac{29}{30}$.

65. $\frac{\frac{4x}{3}+5}{2}+\frac{\frac{2x}{5}-\frac{1}{2}}{6}-\frac{x+\frac{1}{4}}{\frac{1}{5}}=5\frac{13}{30}$.

66. $\dfrac{\frac{1}{8}(x-2)}{\frac{2}{3}}+\dfrac{\frac{3}{4}(6x-7)}{5}=\dfrac{\frac{2x}{3}-1}{\frac{1}{6}}-1\frac{1}{4}.$

67. $\dfrac{x-\frac{1}{2}}{x-\frac{3}{2}}+\dfrac{x-\frac{13}{2}}{x-\frac{15}{2}}=\dfrac{x-\frac{3}{2}}{x-\frac{5}{2}}+\dfrac{x-\frac{11}{2}}{x-\frac{13}{2}}.$

68. $\dfrac{3}{2x-3}-\dfrac{2}{3x-2}=\dfrac{5}{6(x-1)}.$

69. $\dfrac{x}{x+1}-\dfrac{3x}{x+2}=-2.$

70. $\dfrac{x}{x-1}+\dfrac{x-1}{x-2}=2+\dfrac{x+3}{2x^2-6x+4}.$

71. $\cdot2x-3\cdot75x=1-3\cdot8x.$ 72. $\cdot5x+\cdot25x-4\cdot5=\cdot375x.$

73. $\cdot2x-\cdot5x+\cdot1x=\cdot25x-9.$

74. $\cdot625x-\cdot25x+\cdot2x=\cdot5\,(x+2)-\cdot4.$

75. $\dfrac{x+1}{\cdot2}-\dfrac{\cdot3x}{2\cdot5}-7\cdot5x=18\cdot1.$

76. $\dfrac{\cdot2x}{\cdot01}-\dfrac{x+\cdot25}{\cdot5}-\dfrac{1\cdot5x+9\cdot75}{\cdot6}=12\cdot15x.$

77. $\dfrac{\cdot25x}{\cdot3}-\dfrac{\cdot5x-2\cdot8}{2\cdot5}=\dfrac{\cdot3x+\cdot34}{\cdot75}+\dfrac{1\cdot2}{\cdot4}.$

78. $\dfrac{3\cdot3x+\cdot1}{\cdot75x+1}+\dfrac{\cdot2x+\cdot5}{\cdot25x-1}=5\cdot2.$

79. $\cdot25x+\dfrac{\cdot5-\cdot75x}{4}=\dfrac{\cdot375x+\cdot25}{4}.$

80. $\dfrac{\cdot6x-\cdot4}{\cdot4x-1}-3\,(1\cdot4-\cdot2x)=\cdot6x+1\cdot3.$

81. $(x-2)(x-1)+2(x+4)=(x-3)(x+2)+16.$

82. $(2x+5)(x-7)-3(x-5)=(x-3)(x-4)+x(x-9).$

83. $(x-4)^2-3(2x-7)=(x+2)(x-7)-\frac{x}{2}.$

84. $3(x+3)(x-2)=3(x-4)-(2x-1)^2+x(x-3).$

85. $\left(x-\frac{5}{2}\right)\left(x+\frac{3}{2}\right)-(x-5)(x+3)=23\frac{1}{4}.$

86. $\sqrt{x+7}+\sqrt{x}=7.$

87. $\sqrt{2x}-\sqrt{2x-11}=1.$

88. $\sqrt{3x}-\sqrt{3x-5}=1.$

89. $\sqrt{3x+4}+\sqrt{3x-5}=9.$

90. $\sqrt{4x+9}+2\sqrt{x+6}=15.$

91. $\sqrt{7x+2}=\sqrt{7x-5}+1.$

92. $\sqrt{9x+7}-2=3\sqrt{x-1}.$

93. $\sqrt{x+4}+\sqrt{x-4}=4.$

94. $\sqrt{5x+10}-\sqrt{5x}=2.$

95. $\sqrt{4x+9}-2\sqrt{x}=1.$

96. $\sqrt{x+8}-\sqrt{x+3}=\sqrt{4x+9}.$

97. $\frac{1}{x^2-13x+42}-\frac{2}{x^2-6x-7}=\frac{5}{x^2-5x-6}.$

98. $\frac{5}{x^2-2x-15}-\frac{7\frac{1}{2}}{x^2+2x-35}=\frac{1}{x^2+10x+21}.$

99. $\frac{10}{x+20}+\frac{15}{x+5}=\frac{20}{x+10}+\frac{5}{x+15}.$

100. $(3x-1)^2+(4x-2)^2=(5x-3)^2.$

CHAPTER XII.

PROBLEMS.

59. WE have already explained the method by which problems are solved, and have proposed some easy examples: we shall now give the solution of some which present rather more difficulty.

Ex. 1. The length of a rectangle exceeds its breadth by 4 feet; if the breadth be diminished by a foot and the length be increased by 2 feet, its area is not altered: find its length and breadth.

Let x represent the breadth of the rectangle in feet, then $x+4$ is the length.

Since the area is obtained by taking the product of the length and breadth, we have $x(x+4)$ to represent the number of sq. feet in the rectangle.

Also, in the new rectangle, the area will be expressed by

$$(x-1)(x+6).$$

Hence
$$x(x+4)=(x-1)(x+6),$$
$$x^2+4x=x^2+5x-6,$$
$$x=6.$$

The length and breadth are therefore 10 feet and 6 feet. Ans.

Ex. 2. A farmer bought 22 cows at a certain price: had he paid 8 per cent. less he would have purchased one more cow, and would have had £21 to spare: find the price of each cow.

Let x be the price of one cow, in £s; then the price of all the cows was $22x$: now, if the price had been 8 per cent. less, the cost of each in £s would have been $x-\frac{8x}{100}\left(=\frac{23x}{25}\right)$; and in this case there would have been 23 cows:

hence we have

$$22x = 23 \times \frac{23x}{25} + 21,$$

$$550x = 529x + 525,$$

$$21x = 525,$$

$$x = 25.$$

The price of each cow was £25. Ans.

Ex. 3. Find at what times between 7 and 8 o'clock the hour and minute-hands of a watch are at right angles to each other.

The method by which all questions of this kind are solved is to determine the space traversed by the minute-hand since it left the figure XII, and (as it travels twelve times as fast as the hour-hand) to form the equation by writing this space equal to 12 times that traversed by the hour-hand in the same time.

Let x be the number of minute spaces traversed by the hour-hand since 7 o'clock; then as the hour-hand was at that time 35 minute spaces in advance of the minute-hand, and is now only 15 minute spaces from it, the latter will have traversed either $(35 + x - 15)$ or $(35 + x + 15)$ spaces, according as it is after or before the hour-hand. In the former case we have

$$35 + x - 15 = 12x,$$

$$11x = 20,$$

$$x = 1\tfrac{9}{11},$$

so that the hour-hand has moved $1\frac{9}{11}$ spaces, and the minute-hand $21\frac{9}{11}$ spaces; i.e. the time is $7 \cdot 21\frac{9}{11}$.

In the latter case we have

$$35 + x + 15 = 12x,$$

$$11x = 50,$$

$$x = 4\tfrac{6}{11},$$

so that the minute-hand has moved $54\frac{4}{11}$ spaces, i.e. the time is $7 . 54\frac{4}{11}$.

Hence $7 . 21\frac{9}{11}$ and $7 . 54\frac{4}{11}$ are the required times.

Exercise XII.

1. Find the number whose double exceeds its fifth part by 36.

2. What is the number of which the twenty-fifth part is larger by 8 than the thirty-fifth part?

3. Divide 120 into two parts such that the less may be three-fifths of the greater.

4. Divide 52 into two parts, such that the first may exceed twice the second by 4.

5. Divide the number 72 into two parts, such that one part increased by 9 may be equal to twice the other.

6. £500 was divided between A and B, so that one-fifth of A's share was equal to one-third of B's: how much did B receive?

7. Divide a guinea into two parts such that three times one of them shall exceed five times the other by a shilling.

8. A and B have £4. 16*s*. between them, and half of A's money exceeds one-third of B's by half a guinea: how much has B?

9. A has twice as much money as B; if B give A £5, A will have three times as much as B: find what A has.

10. A and B have equal sums of money, A spends 12*s*. and B 5*s*., and now B has twice as much as A. How much had each at first?

11. Find two numbers, one of which is four-fifths of the other, and whose difference is 13.

12. A daughter is half the age of her mother and eleven years ago she was two-fifths of her age: how old was her mother then?

13. A and B have equal sums of money: A spends £750, and B £100, after which B has half as much again as A: how much had each at first?

14. Divide £1800 among A, B, and C, giving B half as much as A, and C £200 less than B.

15. A, B, and C gave £1575 to a charity: B gave £125 more than C, and A gave twice as much as B: how much did A give?

16. A and B have £100 between them; A and C £125; B and C £185: how much has each?

17. Divide 150 into two parts, such that their sum when divided by their difference shall be 25.

18. There is a fraction whose denominator exceeds its numerator by 4; also if the numerator be diminished by 1, and the denominator increased by 1, the fraction becomes equal to $\frac{1}{2}$; find it.

19. Find two numbers whose difference is 25, such that the second divided by the first gives 4 as quotient and 4 as remainder.

20. A farmer has oxen worth £12. 10*s*. each, and sheep worth £2. 5*s*. each, the number of oxen and sheep being 35, and their value £191. 10*s*. Find the number he has of each.

21. A man rows 6 miles an hour in still water. How far can he row up stream and back in 2 hours, the stream flowing a mile an hour?

22. A person bought 4 horses; for the second he gave half as much again as for the first; for the third half as much again as for the second; and for the fourth as much as for the first and third together; he paid £800 for them all. What was the cost of the fourth horse?

23. A man bought some sheep for £94, and having lost seven of them sold a quarter of the remainder at prime cost for £20: how many did he buy?

24. A company of soldiers is formed into a solid square, and the formation is then changed to a hollow square, two deep; it is observed that by this change the front is increased by only 2 men: how many men were there in the company?

25. A quadrangle whose length is 3 yards more than its breadth, is reduced by a covered walk 5 yards wide along two of its adjacent sides, and its area is now found to be 590 sq. yds. less than it was before: find the original length of the quadrangle.

26. At what time between 4 and 5 o'clock are the hands of a watch in a straight line?

27. At what time between 2 and 3 o'clock is the minute-hand of a watch directly over the hour-hand?

28. At what times between 6 and 7 o'clock are the hands of a watch at right angles to each other?

29. How many minutes after 3 o'clock is it when the minute-hand of a watch is exactly 20 minute-spaces before the hour-hand?

30. At what time between 7 and 8 o'clock is the minute-hand of a watch exactly twenty minutes behind the other?

31. The length of a certain rectangle is double its breadth: if the length be diminished by 6 feet, and the breadth increased by 4 feet, the area remains unchanged: find the original length.

32. The perimeter of a rectangle is 28 yards: if 2 yards be taken from its length and added to its breadth, its area is increased by 12 square yards. Find its original breadth.

33. Find two numbers such that the difference of their squares is four times the difference of the numbers, one of the numbers being three times the other.

34. Two trains approach one another with a velocity of 100 miles an hour: and if going in the same direction would separate by 5 miles in half an hour. Find their actual velocities.

35. *A* and *B* start to walk from Cambridge to Ely, at the rate of $3\frac{1}{2}$ and 4 miles an hour respectively. After *B* has gone two-thirds of the way, he waits for 36 minutes, which just allows *A* to catch him up; they then start off simultaneously each at the same pace as before. How far is it from Cambridge to Ely, and how long will *B* arrive at Ely before *A*?

36. *A* has a certain number of sovereigns, *B* the same number of half-crowns. *A* pays *B* £9, and receives back 2*s.* 6*d.*, and then finds that his money is less than *B*'s by 5*s.* How much had each at first?

37. The difference of the squares of two consecutive numbers is 91: find them.

38. *A* sets out for a day's walk; after walking 50 minutes he increases his rate by one-fourth and keeps this up for 2 hours. After a rest he starts again at eight-ninths of his last rate and walks for 6 hours, when he finds he is 35 miles from where he started. What was his rate at first?

39. A man can scull 9 miles an hour with the stream and 3 against it; how far down may he go so as to be back in 6 hours?

40. *A* and *B* enter for a mile race: *A*, who can run the distance in $4\frac{1}{2}$ minutes, gives *B* a start of 20 yards and reaches the winning post four seconds before him: at how many yards from the starting-point will *A* overtake *B*?

41. Find a number such that whether it is divided into two or three equal parts, the continued product of the parts shall be the same.

42. A man walks along a line of railway at the rate of 4 miles an hour. A goods train travelling 30 miles an hour overtakes him. If the train be 208 yards long, how long will it take in passing the man?

43. The first digit of a number exceeds the second by 5, and if the digits be inverted the new number will be three-eighths of the original number; find the number.

44. A person bought 25 yards of carpet; find the price per yard he paid for it when a reduction of 10 per cent. would have given him two yards more of the carpet, and 5*s.* 3*d.* over.

45. A furniture dealer bought 12 chairs, and afterwards 4 tables for the same sum; if the price of the chairs

had been 25 per cent. higher, and the price of the tables 50 per cent. lower, 9 tables would have cost him £1. 10*s.* less than 12 chairs. Find what he paid for each chair.

46. A bill is paid by a number of half-crowns, and twice the same number of shillings. If the bill had been larger by 3 half-crowns, it would have been defrayed by paying as many half-crowns as there were shillings before, and as many shillings as there were half-crowns: find the amount of the bill.

47. *A* runs round a field in a minutes, *B* runs a mile in b minutes: in a race of distance c yards, *A* wins by d yards. Find the length round the field supposing the speed of the runners constant.

48. Divide £900 into two parts such that the simple interest on the first part at $4\frac{1}{2}$ per cent. may exceed that on the second part at $3\frac{1}{2}$ per cent. by 10*s.*

CHAPTER XIII.

Simultaneous Equations.

60. Simultaneous Equations which contain decimal fractions are readily solved by changing the decimals into vulgar fractions and proceeding by the ordinary method.

Ex. 1. Solve $\left.\begin{aligned} 1{\cdot}5x - 2y &= 2\,; \\ \frac{x-y}{{\cdot}4} - \frac{{\cdot}3x}{2} &= 3{\cdot}45. \end{aligned}\right\}$

Change the decimal into vulgar fractions, and simplify; then from the first equation we obtain

$$15x - 20y = 20 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)\,;$$

the second equation becomes

$$10x - 10y - \cdot 6x = 13\cdot 8,$$

$$9\cdot 4x - 10y = 13\cdot 8,$$

$$94x - 100y = 138 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiply (1) by 5 and subtract the result from (2), thus

$$94x - 100y = 138,$$

$$75x - 100y = 100,$$

$$19x = 38,$$

$$x = 2.$$

Substitute in (1); then

$$30 - 20y = 20,$$

$$20y = 10,$$

$$y = \frac{1}{2} = \cdot 5.$$

Hence the values are 2 and $\cdot 5$.

61. When equations are given, in which the unknown letters appear as plain quantities in the denominators only, they must be solved without removing the letters from the denominators, the reciprocals of the letters $\left(\frac{1}{x}, \frac{1}{y}\right)$ being treated as the unknown quantities.

Ex. 2. Solve $\frac{6}{x} - \frac{4}{y} = 2$; $\frac{5}{x} - \frac{6}{y} = 1$.

Leaving x and y in the denominators, multiply the first equation by 5, and the second by 6, so as to make the coefficients of x the same in both; thus

$$\frac{30}{x} - \frac{20}{y} = 10,$$

$$\frac{30}{x} - \frac{36}{y} = 6;$$

then by subtraction

$$\frac{16}{y}=4\,;\quad \therefore \frac{1}{y}=\frac{1}{4} \text{ and } y=4. \quad \text{Ans.}$$

and by substitution

$$\frac{6}{x}-1=2\,;\quad \therefore \frac{1}{x}=\frac{1}{2} \text{ and } x=2. \quad \text{Ans.}$$

Ex. 3. Solve $10x+y=4xy$; $x+\frac{y}{2}=\frac{6xy}{5}$.

Equations of this kind may be reduced to the form in the last example by dividing both sides of the equation by xy; thus we have

$$\frac{10}{y}+\frac{1}{x}=4\,;\quad \frac{1}{y}+\frac{1}{2x}=\frac{6}{5}.$$

Multiplying the second equation by 10, we obtain

$$\frac{10}{y}+\frac{5}{x}=12,$$

whence by subtraction

$$\frac{4}{x}=8, \text{ or } x=\frac{1}{2},$$

and by substitution we find $y=5$.

It will be observed that when the equations are given in the form of this example, they are also satisfied by having each of the letters equal to 0.

62. The methods in the last two sections may be applied in exactly the same way to equations with three unknown letters.

63. Equations are sometimes proposed containing more than three unknown quantities; the first step in solving them is to eliminate one of the letters from all the equations which contain it; this will also make the number of

the equations one less: the same process of removing a letter must be repeated with those that are left; and so on, until there remain only one equation and one unknown letter.

Ex. 4. Solve

$$2w+3x-2y-z=1 \quad \ldots\ldots (1).$$
$$5w+10x-19y-12z=-72 \quad \ldots\ldots (2).$$
$$2x+y+4z=13 \quad \ldots\ldots (3).$$
$$w+4x-2y=1 \quad \ldots\ldots (4).$$

Since w appears in only 3 equations, we remove it first:

Multiply (4) by 2 and combine with (1),

$$2w+8x-4y=2,$$
$$2w+3x-2y-z=1,$$
$$\therefore 5x-2y+z=1. \quad \ldots\ldots (5).$$

Multiply (4) by 5 and combine with (2),

$$5w+20x-10y=5,$$
$$5w+10x-19y-12z=-72,$$
$$\therefore 10x+9y+12z=77 \quad \ldots\ldots (6).$$

We thus have 3 equations (3), (5), (6) with only 3 unknown letters; from these we shall eliminate y:

Multiply (3) by 2 and combine with (5).

$$4x+2y+8z=26,$$
$$5x-2y+z=1,$$
$$\therefore 9x+9z=27,$$
$$\text{or } x+z=3 \quad \ldots\ldots (7).$$

Multiply (3) by 9 and combine with (6).

$$18x+9y+36z=117,$$
$$10x+9y+12z=77,$$
$$\therefore 8x+24z=40,$$
$$\text{or } x+3z=5 \quad \ldots\ldots (8).$$

Now take together (7) and (8), and we find $2z=2$, or $z=1$, and by substitution in (7), $x=2$.

Substitute in (5) these values of x and z; then

$$10-2y+1=1,$$

$$\therefore 2y=10, \text{ or } y=5.$$

Substitute in (4) these values of x and y; then

$$w+8-10=1,$$

$$\therefore w=3.$$

Hence the answers are 3, 2, 5, 1.

EXERCISE XIII.

Solve the following equations:

1. $\frac{x}{2}+\frac{y}{2\frac{1}{2}}=1\frac{3}{4}$; $\frac{x-y}{11}=\frac{x}{1\frac{3}{4}}-\frac{3}{14}$.

2. $\cdot5x+\cdot2y=1\cdot1$; $\cdot25x+\cdot6y=2\cdot05$.

3. $\frac{\cdot4x+2\cdot5y}{2}-\frac{\cdot2x}{\cdot25}=2$; $x-y=1$.

4. $\frac{x+y}{\cdot8}+\frac{y}{\cdot6}=6\cdot5$; $\frac{3x-\cdot3y}{\cdot2}=1$.

5. $\frac{12}{x}-\frac{2}{y}=1$; $\frac{4}{x}+\frac{3}{y}=2\frac{1}{6}$.

6. $\frac{6}{x}-\frac{1}{y}=\cdot4$; $\frac{11}{x}-\frac{1}{2y}=1$.

7. $3x+4y=10xy$; $2x-3y=xy$.

8. $x+8y=\frac{12xy}{5}$; $3x-2y=\frac{7xy}{10}$.

9. $\frac{x}{5}-2y=\frac{xy}{10}$; $\frac{x}{10}\left(3-\frac{y}{2}\right)=5y$.

10. $\frac{4}{x}-\frac{5}{y}=\frac{1}{2}$; $3x-y=\frac{xy}{20}$.

11. $3x+2y+z=10$; $x+y+3z=12$; $x+z=2y$.

12. $\cdot5x+\cdot75y+z=9$; $\cdot2x+\cdot1y=\cdot16z$; $x+y=1\cdot2z$.

13. $2x-\cdot8y=\cdot2$; $\cdot2x+\cdot5z=1\cdot1$; $\cdot375y-\cdot125z=\cdot125$.

14. $\cdot25x+\cdot5y=\cdot75x+z=y-\cdot2z=2$.

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6$; $\frac{2}{x}+\frac{3}{y}+\frac{2}{z}=15$; $\frac{3}{x}+\frac{2}{y}-\frac{5}{z}=7$.

16. $5x-3y=2xy$; $yz+2zx+3xy=9xyz$; $\frac{2}{x}+\frac{1}{z}=\frac{5}{2y}$.

17. $\left.\begin{aligned}&\frac{3}{x}+\frac{4}{y}+\frac{1}{z}=4;\quad yz+zx+xy=\frac{11}{6}xyz;\\&2xz+3yz=2xy.\end{aligned}\right\}$

18. $\left.\begin{aligned}w+x+y+z&=12,\\2w+3x+2y-z&=23,\\3w-2x+4y&=6,\\5w-x-3y&=12.\end{aligned}\right\}$

19. $\left.\begin{aligned}2w-x-y&=7,\\2x-y-z&=4,\\2y-z-w&=-2,\\2z-w+x&=-3.\end{aligned}\right\}$

20. $\left.\begin{aligned}4y+3z&=2yz,\\2z+4w&=3zw,\\4w+3x&=5wx,\\\frac{3}{y}+\frac{2}{x}&=2.\end{aligned}\right\}$

21. $\left.\begin{aligned}3w+2x&=2wx,\\5y&=2z(7y-2),\\8xz&=3x+4z,\\4xy+2yw+wx&=3wxy.\end{aligned}\right\}$

CHAPTER XIV.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

64. Ex. 1. Find a fraction such that if its denominator be increased by 1, it becomes $\frac{1}{6}$; but if its numerator be increased by 1, and its denominator diminished by 1, it becomes $\frac{1}{4}$.

Let $\frac{x}{y}$ represent the fraction,

then $\frac{x}{y+1}=\frac{1}{6}$ .....(1),

and $\frac{x+1}{y-1}=\frac{1}{4}$ .....(2).

These equations become $6x-y=1$,

$4x-y=-5$,

whence $2x=6$; or $x=3$; and $y=17$.

Therefore $\frac{3}{17}$ is the required fraction. Ans.

Ex. 2. There is a number of 3 digits, the middle digit of which is equal to the sum of the other two; also if the first and last digits be transposed, the number is increased by 99, but if the second and last are transposed the number is diminished by 36: find the number.

Let x, y, and z be the digits,

then the number is $100x+10y+z$;

and we have the equations

$$y=x+z \quad \text{..... (1).}$$

$$100z+10y+x=100x+10y+z+99 \quad \text{..... (2).}$$

$$100x+10z+y=100x+10y+z-36 \quad \text{..... (3).}$$

Simplifying, we obtain

$$x-y+z=0,$$
$$x-z=-1,$$
$$y-z=4:$$

whence $x=4, \quad y=9, \quad z=5,$

and the required number is 495. Ans.

Exercise XIV.

1. *A* and *B* have together £570: if *A*'s money were three times what it is, and *B*'s money five times what it is, the sum would be £2350: what is the money of each?

2. Find a fraction such that if 1 be added to its numerator, it becomes $\frac{1}{2}$; but if 1 be added to its denominator, it becomes $\frac{3}{7}$.

3. *A* and *B* in rifle shooting each scored 19; *A* by 3 bulls' eyes and 2 outers, *B* by 1 bull's eye and 7 outers; how much is a bull's eye worth?

4. Find the fraction which becomes $\frac{1}{4}$ when 1 is added to its numerator, but becomes $\frac{1}{5}$ if 1 be added to its denominator.

5. Find two numbers which whether the less be subtracted from the greater, or the greater be divided by the less, have the same result 7.

6. A certain fraction becomes $\frac{1}{2}$ when its numerator is doubled and its denominator increased by 1, but if its numerator is increased by 1 and its denominator diminished by 1, it becomes $\frac{1}{3}$: find the fraction.

7. A and B bought some nuts: if A had bought 5 more, he would have had half as many again as B, and if B had bought 15 less he would have had half what A had; how many did each buy?

8. The sum of two digits is 9, and six times one of the numbers they form is equal to five times the other: find the numbers.

9. Find the number consisting of two digits whose sum is 9, such that if the order of the digits be reversed the number so obtained : the original number :: 7 : 4.

10. Find two numbers in the ratio of 3 to 4, such that if the first be increased by 7 and the second be doubled, they may be in the ratio of 2 to 3.

11. A certain fraction would be doubled by adding 14 to its numerator and 6 to its denominator, and it would be trebled by adding 7 to its numerator and taking 4 from its denominator: find the fraction.

12. A bill of 25 guineas was paid with crowns and half-guineas, and twice the number of half-guineas exceeded three times that of the crowns by 17: how many were there of each?

13. A pound of tea and 3 pounds of sugar cost 4*s.* 6*d.*, but if tea were to rise 20 per cent. in price, and sugar 25 per cent., the cost would be 5*s.* 6*d.*: find the price per pound of each.

14. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by a fourth of its present length, and that of the hind-wheel by a fifth of its present length, the fore-wheel will make 4 revolutions more than the hind-wheel in going 120 yards: find the circumferences of the two wheels.

15. A number consists of three digits whose sum is 10; the middle digit is equal to the sum of the other two; also if 99 be added to the number, the digits are reversed: find the number.

16. A greyhound courses a hare; the greyhound takes 2 strides per second and the hare $2\frac{1}{4}$; the greyhound starts at a distance equal to 4 of his own strides behind the hare, and catches it in 2 minutes and 40 seconds; find the ratio of the lengths of strides of the two.

17. Find the length and breadth of a rectangle such that if 3 feet were taken from the length and added to the breadth its area would be increased by 6 square feet, but if 3 feet were taken from its breadth and 4 feet added to its length its area would be diminished by 20 square feet.

18. Find a number of four digits whose sum is 16 and such that either of the two middle digits is the difference between the first and last; also if the second digit be transposed with the last, the number is increased by 297.

19. Determine two pairs of consecutive numbers such that the product of the greatest and least is less by 14 than the product of the other two, and that the quotient of the greatest divided by the least is 4 with a remainder 3.

20. A number of posts are placed at equal intervals along a road; if the intervals had each been 2 feet less, two more posts would have been required, but if they had been 4 feet less, it would have taken five more posts: find the number of posts and the distance between the extreme posts.

21. A man bought 7 photographs and 12 prints for £50; if he had paid £1 more, he might have had 7 prints and 15 photographs. Find the price of a print.

22. Two mixtures of wine and water contain respectively $\frac{1}{5}$ and $\frac{2}{3}$ wine. How much of each must be taken to form 22 gallons of a mixture in which the wine is to the water as 5 to 6?

23. A miller has two kinds of wheat, one worth 7 shillings, and the other 6 shillings a bushel; he wishes to make a mixture worth 6*s.* 8*d.* per bushel: how must he do it?

24. At a certain time between 10 and 11 o'clock the hour and minute hands of a watch have the same places which the minute and hour hands had at a certain time after 6 o'clock: find the two times.

25. A boat's crew row 9 miles with the tide in $\frac{3}{4}$ of an hour, and when the tide is flowing at half its former rate the same crew row 9 miles against the tide in an hour and a half. Required the rate of the stronger tide and the rate at which the crew will row in still water.

26. A plantation of trees is arranged in rows, some containing one tree less and others four less than the longest rows; and the number of complete rows exceeds by 5 those that have one less and by 10 those that have four

less. There are 220 trees more in the longest rows than in the shortest rows, and the total number of trees in the plantation is exactly three times the number in the rows of middle length. How many trees are there in the plantation?

27. *A*'s income is half as much again as *B*'s while his expenditure is twice as great as *B*'s expenditure; *A* spends £60 more than his income, and *B* £60 less than his: find their incomes.

28. *A* invested some money so as to produce 5 per cent., while *B* invested at 4 per cent. a sum sufficient to produce the same return as *A* got. If *A* had invested £420 more than he did, his interest would have been exactly one-tenth of *B*'s investment. What did each invest?

29. *A* purchased a certain number of shares for £2500; afterwards when the price had risen 50 per cent., he bought some more shares for £4500: if at the time of his first purchase the price had been 20 per cent. more than it was, the sum he would have paid for each share would have been the same as the number of shares he afterwards bought: how many shares did he buy each time?

30. A certain crew who row 40 strokes per minute start at a distance equivalent to four of their own strokes behind another crew who row 45 strokes to the minute; in 8 minutes the former succeed in bumping the latter: find the ratio between the lengths of the strokes in the two boats.

31. A tradesman buys eggs at a distance at the rate of 5 for 3*d.*; some are broken in the transit, and having sold the rest at a penny a piece, he finds he has gained

on the whole transaction just the price of the broken eggs. Had all the eggs come safe, he might have sold them at the rate of 5 for 4*d.*, and gained thereby 10*d.* more than he did gain. How many eggs were broken?

32. A person walking along a road in a thick fog, meets one waggon, and overtakes another which is travelling at the same rate as the former: shew that if a be the greatest distance to which he can see, and b, b' be the distances which he walks between the times of his first seeing and passing the waggons, then

$$\frac{2}{a} = \frac{1}{b} + \frac{1}{b'}.$$

CHAPTER XV.

EQUATIONS WITH LITERAL COEFFICIENTS.

65. EQUATIONS sometimes involve not only the unknown letters (x, y, z...) but also the letters (a, b, c,...); the required value of the unknown letters will then generally be in terms of these other letters.

The same methods of solution apply to these equations as have been employed in those with numerical coefficients; after removing the fractions (if any), we place all terms involving the unknown letter on the left side of the equation, and the others on the right side; the unknown letter must then be written outside a bracket, with its various coefficients in the bracket; and the required value is obtained by dividing the expression on the right hand by the coefficient of the unknown letter.

Ex. 1. Solve the equation

$$a^2(x-1) - b^2(x+1) = 2ab(1-x) - x(a^2+b^2).$$

Remove the brackets and we have

$$a^2x - a^2 - b^2x - b^2 = 2ab - 2abx - a^2x - b^2x,$$

$$2a^2x + 2abx = a^2 + 2ab + b^2,$$

$$2ax(a+b) = (a+b)^2,$$

$$x = \frac{(a+b)^2}{2a(a+b)} = \frac{a+b}{2a}. \quad \text{Ans.}$$

Ex. 2. Solve

$$\frac{x+5a}{x+4a} + \frac{x-a}{x-2a} = \frac{x+4a}{x+3a} + \frac{x}{x-a}.$$

The equation may be written

$$\frac{x+5a}{x+4a} - \frac{x+4a}{x+3a} = \frac{x}{x-a} - \frac{x-a}{x-2a},$$

$$\frac{-a^2}{(x+4a)(x+3a)} = \frac{-a^2}{(x-a)(x-2a)},$$

$$(x+4a)(x+3a) = (x-a)(x-2a),$$

$$x^2 + 7ax + 12a^2 = x^2 - 3ax + 2a^2,$$

$$10ax = -10a^2,$$

$$x = -a. \quad \text{Ans.}$$

66. When simultaneous equations with literal coefficients are proposed for solution, the method is the same as for ordinary simultaneous equations, except that in equalising the coefficients we have generally to multiply the equations by letters instead of numbers.

Ex. 3. Solve the equations

$$\frac{x}{a-b} + \frac{y}{b} = 3 + \frac{a}{b}; \quad \frac{x}{a} - \frac{y}{a+b} = 1 - \frac{2b}{a}.$$

Removing the fractions, we have

$$bx + (a-b)y = (a+3b)(a-b) \ldots\ldots\ldots\ldots (1),$$

$$(a+b)x - ay = (a-2b)(a+b) \ldots\ldots\ldots\ldots (2).$$

Multiply (1) by $a+b$ and (2) by b; then

$$bx(a+b)+(a^2-b^2)\,y=a^3+3a^2b-ab^2-3b^3,$$

$$bx(a+b)-aby=a^2b-ab^2-2b^3.$$

By subtraction $(a^2+ab-b^2)\,y=a^3+2a^2b-b^3,$

$$y=a+b.$$

Substitute in the 2nd equation, and we have

$$\frac{x}{a}-1=1-\frac{2b}{a},$$

$$\frac{x}{a}=2-\frac{2b}{a}=\frac{2a-2b}{a},$$

$$x=2(a-b).$$

Hence the values are $x=2(a-b)$; $y=a+b$.

Ex. 4. Solve

$$\frac{a+b}{y}-\frac{a-b}{z}=\frac{a}{b}; \qquad \frac{1}{z}-\frac{1}{x}=\frac{b}{a^2-ab}; \qquad \frac{a}{x}+\frac{b}{y}=2.$$

In equations like these, where the unknown letters appear only in the denominators and are of the first power, it is generally best to leave them as fractions, considering the reciprocals as the unknown quantities instead of the letters themselves (see Art. 61).

In this case, take together the 2nd and 3rd equations, multiplying the second by a: thus we have

$$\frac{a}{z}-\frac{a}{x}=\frac{b}{a-b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\frac{a}{x}+\frac{b}{y}=2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

whence by addition $$\frac{b}{y}+\frac{a}{z}=\frac{2a-b}{a-b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Now combine the first equation with (4), so as to eliminate z;

$$\frac{(a+b)\,a}{y} - \frac{(a-b)\,a}{z} = \frac{a^2}{b},$$

$$\frac{(a-b)\,b}{y} + \frac{(a-b)\,a}{z} = 2a - b.$$

Hence by addition

$$\frac{(a+b)\,a}{y} + \frac{(a-b)\,b}{y} = \frac{a^2}{b} + 2a - b,$$

$$\frac{a^2+2ab-b^2}{y} = \frac{a^2+2ab-b^2}{b},$$

$$\frac{1}{y} = \frac{1}{b} \quad \text{or} \quad y = b.$$

Substituting in (1), we have $\frac{a+b}{b} - \frac{a-b}{z} = \frac{a}{b}$;

whence $\frac{a-b}{z} = 1$, or $z = a - b$.

Substituting in (3), we have $\frac{a}{x} + 1 = 2$,

whence $x = a$.

Hence the values are a, b, $a-b$. Ans.

Exercise XV.

Solve the following equations:

1. $(a+b)\,x - a^2 = b^2$.
2. $(a-b)\,x + 2ab = (a+b)^2$.
3. $ax + bx + b^2 = a^2$.
4. $ax - bx - b^2 = a^2$.
5. $ax - a^2 = 2\,(bx - 2b^2)$.
6. $cx - d^2 = dx - 2cd + c^2$.
7. $4ax + 4ac - c^2 = 2cx + 4a^2$.
8. $a(x-a) = (b-c)\,x - b^2$.
9. $x\,(a-b) + c^2 = cx + (a-b)^2$.

10. $2x(a+b)-3ab=a(2x-b)-b^2$.

11. $\frac{x}{a}+\frac{x}{b}=1$.

12. $\frac{x}{a}-2=\frac{x}{b}$.

13. $x+b+\frac{a(x-a)}{b}=0$.

14. $\frac{x-c}{a}-\frac{x-a}{b}=\frac{1}{a}(b-c)$.

15. $(x-a)^2-(x-b)^2=(a-b)^2$.

16. $2a(x-c)+(x-a)^2=(x-c)^2$.

17. $\frac{(x-a)^2}{x^2}+\frac{(a-b)^2}{c^2}=1+\frac{a^2}{x^2}$.

18. $\frac{1}{2}(x-a)(x-b)=\frac{1}{3}(x-a)^2+\frac{1}{6}(x-b)^2$.

19. $\frac{(x-b)^2}{a}+\frac{(x+a)^2}{b}=\frac{(a+b)x^2}{ab}$.

20. $\frac{1}{x-a+b}+\frac{1}{x+a-b}=\frac{2}{x-a-b}$.

21. $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{3x^2}{(x-a)(x-b)(x-c)}$.

22. $x(x-a)+2(x-a)(x-b)-3(x-b)(x-c)=0$.

23. $\frac{x-a}{x-2a}-\frac{x-2a}{x-3a}=\frac{x-7a}{x-8a}-\frac{x-8a}{x-9a}$.

24. $\frac{x+a}{x-a}+\frac{x-8a}{x-10a}=\frac{x-a}{x-3a}+\frac{x-6a}{x-8a}$.

25. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

26. $bx-\frac{x}{b}=a+\frac{1}{a}-x-b$.

27. $\frac{x}{a}+\frac{x}{b}=\frac{a}{b}-\frac{b}{a}$.

28. $\frac{ax-b}{c}+a=\frac{x+ac}{c}$.

29. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}$.

30. $\frac{x-a}{b}+\frac{x-b}{c}=\frac{x}{a}+\frac{a-b-c}{c}$.

31. $\sqrt{x+3a^2}-\sqrt{x}=a$. 32. $\sqrt{x-a}-\sqrt{x-b}=\sqrt{b}-\sqrt{a}$.

33. $\sqrt{ax-2ab}+\sqrt{ax+b^2}=2a+b$.

34. $\sqrt{x+a}=1+\sqrt{x-a}$.

35. $\frac{a-c}{2b+x}+\frac{b-c}{2a+x}=\frac{a+b-2c}{a+b+x}$.

36. $ax+by=ab$; $x+y=a$. 37. $ax-by=ab$; $x-y=b$.

38. $ax+by=c^2$; $x-y=c$. 39. $x-y=a$; $ax+by=0$.

40. $ax-by=c$; $bx+ay=d$.

41. $\frac{1}{2}(x+a)-\frac{1}{3}(y+b)=0$; $\frac{x}{a}=\frac{y}{b}$.

42. $\frac{x}{c}+\frac{y}{d}=1$; $x-y=c-d$.

43. $\frac{ax}{c}+\frac{by}{d}=a+b$; $\frac{x}{3c}+\frac{y}{6d}=\frac{1}{2}$.

44. $(a+b)x-(a-b)y=2b$; $x+y=2$.

45. $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$; $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}$.

46. $\frac{a}{x}+\frac{b}{y}=2$; $\frac{b}{x}-\frac{a}{y}=\frac{b^2-a^2}{ab}$.

47. $\frac{a}{x}+\frac{b}{y}=c$; $\frac{b}{x}+\frac{a}{y}=d$.

48. $ax-cy=c^2$; $\frac{x}{a}+\frac{y}{c}=2$.

49. $x+y+2z=3a$; $2x-y+z=6b$; $3x+2y-z=6a$.

50. $x+2y=a$; $2x+4z=3a$; $4y+5z=5a$.

51. $ax+by=ab$; $2by+3cz=2ac$; $4cz+5ax=10bc$.

52. $$\left.\begin{array}{c} lx+my+nz=mn+nl+lm, \\ x+y+z=l+m+n, \\ (m-n)\,x+(n-l)\,y+(l-m)\,z=0. \end{array}\right\}$$

CHAPTER XVI.

QUADRATIC EQUATIONS.

67. WHEN a quadratic equation is to be solved in the ordinary manner, it must be simplified until it has for its first term the square of the unknown quantity with a positive sign and with unity as its coefficient; and in the second place the term involving the first power of the unknown quantity: on the other side of the equation must be the plain term. Then complete the equation by squaring half the coefficient of the second term, and adding this square to both sides of the equation. Now take the square root of both sides and solve the equation thus obtained as in simple equations. (See Part I. Chap. XVII.)

Ex. 1. Solve the equation

$$\frac{x+1}{3(x-3)}+\frac{x-5}{6+x-x^2}=\frac{9+x}{2(x+2)}.$$

First, arrange the terms so as to have descending powers of x in each expression:

$$\frac{x+1}{3(x-3)}-\frac{x-5}{x^2-x-6}=\frac{x+9}{2(x+2)},$$

The L. C. D. is $6(x-3)(x+2)$.

Hence the equation becomes

$$2(x+2)(x+1)-6(x-5)=3(x-3)(x+9),$$

$$2x^2+6x+4-6x+30=3x^2+18x-81,$$

$$x^2+18x=115,$$
$$x^2+18x+81=196,$$
$$x+9=\pm 14,$$
$$\therefore \quad x=5 \text{ or } -28. \quad \text{Ans.}$$

68. When an equation has been reduced to its simplest form, it may be solved more quickly by writing all the significant terms on one side (with zero on the other), and resolving this expression into its factors: since the product of the factors is zero, the equation must be satisfied by either factor being made equal to zero, and thus we shall at once obtain the required values.

Ex. 2. Solve $\dfrac{x-2}{x+2}+\dfrac{x}{2x+3}=\dfrac{8}{15}$.

The equation may be reduced to the form

$$29x^2-41x-138=0.$$

Resolve into factors and we obtain

$$(x-3)(29x+46)=0.$$

Hence we may have either

$$x-3=0, \quad \text{or } 29x+46=0;$$
$$\therefore x=3, \quad \text{or } -\frac{46}{29}. \quad \text{Ans.}$$

69. The student will find that there are many artifices which will assist him in the solution of quadratics, but familiarity with them can only be obtained by considerable practice: the following examples will suggest to him some of those most frequently employed.

Ex. 3. Solve $4x-\dfrac{10}{x}=6.$

Since 2 is a factor in all the terms, we divide by 2, and thus have $2x-\dfrac{5}{x}=3.$

Now multiply by x; $2x^2-5=3x$,

$$2x^2-3x-5=0,$$

$$(2x-5)(x+1)=0;$$

$$\therefore x=2\tfrac{1}{2}, \text{ or } -1. \quad \text{Ans.}$$

Ex. 4. Solve $\dfrac{2x+5}{x-2}-\dfrac{4x+1}{2(x+1)}=3\tfrac{1}{4}$.

Reducing the fractions to mixed quantities, we have

$$2+\frac{9}{x-2}-\left(2-\frac{3}{2(x+1)}\right)=3\tfrac{1}{4}.$$

$$\frac{9}{x-2}+\frac{3}{2(x+1)}=\frac{13}{4},$$

$$36(x+1)+6(x-2)=13(x-2)(x+1),$$

$$42x+24=13x^2-13x-26,$$

$$13x^2-55x-50=0,$$

$$(13x+10)(x-5)=0;$$

$$\therefore x=5, \quad \text{or } -\frac{10}{13}. \quad \text{Ans.}$$

Ex. 5. Solve $2x^2+2x+\sqrt{4x^2-20x+25}=8x+1$.

In any equation involving a root as this does, the student must first examine the root to see if the expression under it is a perfect square; if it be so, he must take its square root, and proceed to simplify the equation in the usual manner. In this example we have

$$2x^2+2x+2x-5=8x+1,$$

$$2x^2-4x-6=0,$$

$$x^2-2x-3=0,$$

$$(x-3)(x+1)=0,$$

$$x=3, \quad \text{or } -1. \quad \text{Ans.}$$

Ex. 6. Solve $4x^2-10x-36-2\sqrt{2x^2-7x+14}=4(x+50)$.

By transposition we have

$$4x^2-14x-2\sqrt{2x^2-7x+14}=236,$$

$$2x^2-7x-\sqrt{2x^2-7x+14}=118.$$

The terms at the beginning of the equation must now be made the same as those under the root; we therefore add 14 to both sides of the equation,

$$2x^2-7x+14-\sqrt{2x^2-7x+14}=132.$$

Let $2x^2-7x+14=z^2$, and the equation becomes

$$z^2-z=132,$$

from which $z=12$.

Hence, $2x^2-7x+14\,(=z^2)=144,$

$$2x^2-7x-130=0,$$

$$(x-10)(2x+13)=0,$$

$$x=10, \quad \text{or} \quad -6\tfrac{1}{2}.$$

Ex. 7. Solve $\sqrt{3x+7}+\sqrt{2x+3}=7$.

In this equation we might square both sides just as they stand, but when there are only two roots, it is generally better to separate them by removing one of them to the other side.

Thus $\sqrt{3x+7}=7-\sqrt{2x+3}$.

Square both sides; $3x+7=49-14\sqrt{2x+3}+2x+3$,

transpose so as to leave the root alone on one side,

$$x-45=-14\sqrt{2x+3},$$

again square $x^2-90x+2025=196\,(2x+3)$

$$=392x+588,$$

$$x^2-482x+1437=0,$$

$$(x-3)(x-479)=0,$$

$$\therefore x=3, \quad \text{or } 479.$$

If these values be substituted in the original equation, it will be found that the latter value requires that the negative sign be given to the second root: hence we give as the solution

$$x=3.$$

Ex. 8. Solve $\sqrt{2x^2+7x+3}-\sqrt{2x^2+5x-2}=1.$

This equation may be solved in the same manner as the last, but it is better to proceed as follows:

we know that $2x^2+7x+3-(2x^2+5x-2)=2x+5,$

now divide each side of this equality by the corresponding expression of the given equation: then

$$\sqrt{2x^2+7x+3}+\sqrt{2x^2+5x-2}=2x+5,$$

add this to the given equation, and we have

$$2\sqrt{2x^2+7x+3}=2x+6,$$

$$\sqrt{2x^2+7x+3}=x+3,$$

$$2x^2+7x+3=x^2+6x+9,$$

$$x^2+x-6=0,$$

$$(x+3)(x-2)=0,$$

$$x=-3, \quad \text{or } 2.$$

The negative value will be found inadmissible, and we have $x=2.$ Ans.

Ex. 9. Solve $\dfrac{\sqrt{x+5}-\sqrt{x}}{\sqrt{x+5}+\sqrt{x}}=\dfrac{\sqrt{2x+1}-2}{\sqrt{2x+1}+2}.$

The solution of this depends upon the proposition, that if

$$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a+b}{a-b}=\frac{c+d}{c-d};$$

or in other words, if two fractions are equal to each other, we may form a new equation by taking the sum of the numerator and denominator on each side for new numerators, and taking their difference for new denominators.

Thus we obtain $$\frac{2\sqrt{x+5}}{2\sqrt{x}}=\frac{2\sqrt{2x+1}}{4},$$

$$\frac{x+5}{x}=\frac{2x+1}{4},$$

$$4x+20=2x^2+x,$$

$$2x^2-3x-20=0,$$

whence $$x=4. \quad \text{Ans.}$$

Ex. 10. Solve $$x^3=\frac{16}{x^3}+6.$$

Remove fractions and transpose; then

$$x^6-6x^3=16.$$

Since the first term is the square of the unknown quantity in the second, we can proceed by the ordinary method of solving a quadratic, x^3 being treated as the unknown quantity;

$$x^6-6x^3+9=25,$$

$$x^3-3=\pm 5,$$

$$\therefore x^3=8, \quad \text{or} \quad -2.$$

Hence the only real value of $x=\sqrt[3]{8}=2$. Ans.

Ex. 11. Solve $$2x^2-\frac{1}{x^{\frac{1}{2}}}=x^{\frac{5}{4}}.$$

Multiply by $x^{\frac{1}{2}}$; then $$2x^{\frac{5}{2}}-1=x^{\frac{5}{4}},$$

$$x^{\frac{5}{2}}-\frac{x^{\frac{5}{4}}}{2}=\frac{1}{2}.$$

Let $x^{\frac{5}{4}}=z$, then $$x^{\frac{5}{2}}=z^2,$$

hence $$z^2-\frac{z}{2}=\frac{1}{2},$$

whence $$z=1,$$

$$\therefore x^{\frac{5}{4}}=1, \quad \text{and} \quad x=1. \quad \text{Ans.}$$

Ex. 12. Solve $3^{x+2}-9^x+486=0$,

$3^{x+2}=3^x\times 3^2=9\,.\,3^x$,

$9^x=(3^2)^x=3^{2x}$.

By substituting these values, we have

$$9\,.\,3^x-3^{2x}=-486,$$

$$3^{2x}-9\,.\,3^x=486.$$

Let $3^x=z$; then $z^2-9z=486$,

whence $z=27$ or -18,

$\therefore\ 3^x=27$,

but we know that $3^3=27$;

$\therefore\ 3^x=3^3$,

$\therefore\ x=3$. Ans.

Exercise XVI.

Solve the following quadratic equations:

1. $3x^2-18x=48$.
2. $2x^2=36x-90$.
3. $x^2=3x+10$.
4. $x^2-\frac{x}{6}=8\frac{1}{3}$.
5. $5x^2-3x+\frac{2}{5}=0$.
6. $12x^2-2x=10$.
7. $x+\frac{4}{x}=4$.
8. $3x=\frac{6}{x}+10\frac{1}{2}$.
9. $x-\frac{10}{x}-3=0$.
10. $x-9\frac{1}{2}=\frac{5}{x}$.
11. $x=\frac{8}{x-2}$.
12. $x-\frac{20}{x-5}=6$.

13. $x-7=\frac{4}{x-4}$.

14. $x-3=\frac{3}{2x-7}+1$.

15. $3x-2=\frac{2x}{x-2}+x+7$.

16. $\frac{15}{3x-25}=x-7$.

17. $(2x+1)(x+8)=27$.

18. $3(2x+1)(3x-2)=4$.

19. $2(x+4)(4x-9)=75$.

20. $2x(x+2)=(3x-2)^2$.

21. $2x(x+4)-(x+2)(x+3)=2(3x+2)$.

22. $3(x+4)^2-2(x-1)(x+3)=2(2x^2+1)$.

23. $\frac{12}{x+2}+3=\frac{10}{x-2}$.

24. $\frac{4}{x+1}+1=\frac{10}{x+2}$.

25. $\frac{15}{2x}-1=\frac{1}{x-3}$.

26. $\frac{6}{x+5}-\frac{2}{x+4}=1$.

27. $\frac{10}{x-3}+\frac{12}{x+4}=3$.

28. $\frac{3}{2x-3}+\frac{8}{x-1}=5$.

29. $\frac{6}{x+4}-\frac{21}{2x-7}=4\frac{1}{2}$.

30. $\frac{3}{1+2x}-\frac{6}{1-2x}=-5$.

31. $\frac{8}{3x+5}-\frac{10}{2x-1}=\frac{1}{6}$.

32. $\frac{2}{6x-1}+\frac{2}{2+3x}=1\frac{1}{4}$.

33. $\frac{7}{4x-3}-\frac{4}{2x-3}+1=0$.

34. $\frac{5}{3x+2}-\frac{3}{x+1}=\frac{1}{2}$.

35. $\frac{3}{2(x+1)}+\frac{5}{2x}=3\frac{1}{4}$.

36. $\frac{2}{3(x+4)}-\frac{3}{2x}=1\frac{1}{6}$.

37. $\frac{6}{5(x-1)}-\frac{8}{5(x+1)}=\frac{2}{3}$.

38. $\frac{5}{6(x-2)}-\frac{7}{3(x+1)}=\frac{1}{4}$.

39. $\frac{1}{x-7}+\frac{2}{3(x-9)}-1=0$.

40. $\frac{2}{3(x-4)}=\frac{5}{4(x-2)}+\frac{1}{4}$.

41. $\frac{2x+1}{x-3}+\frac{4x+2}{2x-5}=15$.

42. $\frac{6x+5}{3x+4}+\frac{2x+5}{2x+3}=2$.

43. $\frac{7x+10}{x+4}-\frac{4x-7}{2x-1}+5=0$.

44. $\frac{3x+5}{x-1}-\frac{2x+3}{x+1}=4\frac{3}{4}$.

45. $\frac{3}{x-1}-\frac{x-4}{x-3}=1$.

46. $\frac{x+1}{x-3}-\frac{x+8}{x+5}=3\frac{2}{3}$.

47. $\frac{3x-7}{2x-5}-\frac{2x}{3x-7}=\frac{1}{15}$.

48. $\frac{x+2}{3x+7}-\frac{x+4}{4x+9}=\frac{5}{6}$.

49. $\frac{2x-3}{4x-1}=\frac{1-5x}{2x+3}-1\frac{5}{8}$.

50. $\frac{5x+2}{2x+5}-\frac{1}{20}=\frac{2x+5}{3(x+2)}$.

51. $\frac{2x-1}{3x+1}+\frac{2}{x-5}=\frac{5}{7}$.

52. $\frac{4x}{9}+\frac{x-5}{x+3}=\frac{4x+7}{19}$.

53. $x^2-6x+12+\sqrt{x^2+6x+9}=5$.

54. $2x^2-9-\sqrt{4x^2+4x+1}=3x-7$.

55. $3x^2-7-2\sqrt{x^2-6x+9}=3+9x$.

56. $7x^2-x-5\sqrt{x^2+4x+4}=4x+7$.

57. $x^2+5x-10+2\sqrt{x^2+5x+2}=12$.

58. $x^2-4x+2\sqrt{x^2-4x+7}=1$.

59. $x^2-4x+6+4\sqrt{x^2-7x+11}=3x$.

60. $2x^2-2x+6\sqrt{x^2-x+7}=22$.

61. $3x-2\sqrt{x^2+3x+9}=26-x^2$.

62. $2x^2+7x-31=x+\sqrt{x^2+3x+7}$.

63. $x^2-3\sqrt{x^2-3x+6}=3x-2$.

64. $2x^2-x+14-11\sqrt{2x^2-3x+16}+30=2(x-1)$.

65. $\sqrt{3x^2-7x+9}+3(x^2+3)=7(x+8)$.

66. $\sqrt{x^2+8x}=3$.

67. $\sqrt{4x+9}+4x=21$.

68. $2x-3\sqrt{x-8}=25$.

69. $\sqrt{2x+3}-\sqrt{x-7}=3$.

70. $\sqrt{x+2}+\sqrt{3x+4}=8$.

71. $\sqrt{x+1}+\sqrt{2x}=7$.

72. $\sqrt{2x-3}+\sqrt{4x+1}=\sqrt{6x+28}$.

73. $\dfrac{\sqrt{2x^2+1}-\sqrt{x^2-3}}{\sqrt{2x^2+1}+\sqrt{x^2-3}}=\dfrac{1}{2}$.

74. $\dfrac{3x+2\sqrt{x^2-5}}{3x-2\sqrt{x^2-5}}=2\frac{2}{5}$.

75. $\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

76. $\dfrac{5x+3\sqrt{2x^2+9}}{5x-3\sqrt{2x^2+9}}=19$.

77. $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\dfrac{x}{a-x}$.

78. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=4\sqrt{2}$.

79. $\dfrac{1}{m-\sqrt{m^2-x^2}}-\dfrac{1}{m+\sqrt{m^2-x^2}}=\dfrac{m}{x^2}$.

80. $\sqrt{2x^2+7}-\sqrt{2x^2-x+1}=1$.

81. $\sqrt{4x^2+6x-9}-2\sqrt{x^2-x-4}=3$.

82. $\sqrt{x^2+3x}+\sqrt{x^2+x+2}=4$.

83. $3\sqrt{x^2+5}-\sqrt{9x^2+4x+5}=2$.

84. $x^4+7x^3=44$.

85. $x^6-56=x^3$.

86. $x^3=9x^{\frac{3}{2}}-8$.

87. $3x=4x^{\frac{1}{2}}+15$.

88. $x^{\frac{1}{5}} - \frac{6}{x^{\frac{1}{5}}} + 1 = 0.$

89. $6x^{-2} = 5x^{-1} = 1.$

90. $x^{-1} + 3 = 4x^{-\frac{1}{2}}.$

91. $x^{-\frac{3}{2}} + 8 = 9x^{-\frac{3}{4}}.$

92. $\frac{2x(a-x)}{3a-2x} = \frac{a}{4}.$

93. $\frac{mx}{mx+a} + \frac{nx}{nx+a} = \frac{2px}{px+a}.$

94. $\frac{(x+b)^2}{a^2} - \frac{(x+a)^2}{b^2} = \frac{b^4-a^4}{a^2b^2}.$

95. $\left(\frac{x-a+b}{x+a-b}\right)^2 = \left(1-\frac{2x}{a}\right)^2 + \left(1+\frac{2x}{b}\right)^2 - 9.$

96. $2^{2x} - 2^{x+2} = 32.$

97. $3^{2x} - 2 \times 3^{x+2} + 81 = 0.$

98. $4^x - 3 \times 2^{x+3} + 80 = 0.$

99. $3^{x+1} + 9^x = 810.$

100. $9^x - 3^{x+1} = 6318.$

101. $\frac{x+3}{x-2} + \frac{5x-3}{5x-6} = 7\frac{1}{3}.$

102. $(4x-3)(6x+7) + 41(x-2) = (3x+2)(7x+12) - 97.$

103. $x^3 + x^2 = 4x + 4.$

104. $\frac{x^3+1}{x+1} + \frac{x^4-1}{x^2+1} = 6.$

105. $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}.$

106. $\sqrt{2x+1} + 2\sqrt{x} = \frac{21}{\sqrt{2x+1}}.$

107. $\frac{3}{x^2-2x+3} - \frac{2}{x^2-2x+2} = \frac{1}{10}.$

108. $x^3 + 8x^2 - 9x = 0.$

109. $x^3 - 8 = 4(x-2)(x+7).$

110. $x^4 - 6x^3 + 10x^2 - 3x = 2.$

CHAPTER XVII.

THEORY OF QUADRATIC EQUATIONS.

70. *To shew that a quadratic equation cannot have more than two roots.*

Let $ax^2+bx+c=0$ be a quadratic equation, and if possible let it have three roots α, β, γ.

Then substituting these values for x, we have

$$a\alpha^2+b\alpha+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$a\beta^2+b\beta+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a\gamma^2+b\gamma+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By subtracting (2) from (1),

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0,$$

and dividing by $\alpha-\beta$ (which by the supposition is not zero),

$$a(\alpha+\beta)+b=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Similarly by subtracting (3) from (1) and dividing by $\alpha-\gamma$,

$$a(\alpha+\gamma)+b=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Now subtract (5) from (4), and we obtain

$$a(\beta-\gamma)=0.$$

But a is by supposition not zero; hence we have

$$\beta-\gamma=0, \text{ or } \beta=\gamma,$$

i.e. the equation has not three roots.

71. When the roots of an equation are given, the equation itself may be formed as follows: suppose a and b

are given as the values of x in an equation; then $x=a$ or b;

$$\therefore\ x-a=0,\quad \text{or}\quad x-b=0,$$

if then we write $(x-a)(x-b)=0$ we evidently have a quadratic equation which is satisfied either by having $x-a=0$ or $x-b=0$, i.e. $x=a$, or $x=b$: hence the required equation is

$$(x-a)(x-b)=0,$$

or

$$x^2-ax-bx+ab=0.$$

Ex. 1. Form the equation whose roots are 6 and -7.

We have $x=6$ and $x=-7$;

$$\therefore\ x-6=0 \text{ and } x+7=0.$$

Hence the required equation is $(x-6)(x+7)=0$,

$$\text{or}\quad x^2+x-42=0.$$

Ex. 2. Form the equation whose roots are

$$6,\ -1,\ \frac{1}{2},\ \text{and } 0.$$

We have $x=6$, $x=-1$, $x=\frac{1}{2}$, or $x=0$,

hence the equation must be

$$x(x-6)(x+1)(2x-1)=0,$$

$$2x^4-22x^3-14x^2+12x=0.\quad \text{Ans.}$$

72. *In any quadratic equation* $x^2+px+q=0$ *the sum of the roots is equal to* $-p$, *and their product is equal to* q.

The equation may be written

$$x^2+px=-q,$$

$$x^2+px+\frac{p^2}{4}=\frac{p^2-4q}{4},$$

$$x+\frac{p}{2}=\pm\frac{1}{2}\sqrt{p^2-4q},$$

$$x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

Let α and β be the roots, then

$$\alpha=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

$$\beta=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q};$$

$$\therefore\ \alpha+\beta=-p,$$

and $$\alpha\beta=\frac{p^2}{4}-\frac{p^2-4q}{4}=q.$$

73. *A quadratic equation* $x^2+px+q=0$ *will have two real and unequal roots, two real but equal roots, or two impossible roots according as* p^2 *is* $>$, $=$, *or* $<4q$.

If α and β be the roots, we have by the last section

$$\alpha=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

$$\beta=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q},$$

hence if $p^2>4q$ the quantity under the root is positive, and the values of α and β are real and unequal:

if $p^2=4q$, then $\alpha=-\frac{p}{2}$, and $\beta=-\frac{p}{2}$,

i. e. the roots are real but equal:

if $p^2<4q$, the quantity under the root is negative, and both roots are impossible.

The above shews that if one root of a quadratic equation is real, the other must also be real.

Exercise XVII.

Form equations with the following roots:

1. $2; 4.$ 2. $7; 9.$ 3. $8; \frac{1}{2}.$

4. $\frac{3}{4}; 100.$ 5. $-2; -7.$ 6. $-1; -8.$

7. $-2; \frac{2}{5}.$ 8. $-\frac{1}{5}; -15.$ 9. $3; 4; 1.$

10. $2; -2; 3.$ 11. $7; -1; \frac{3}{2}.$ 12. $0; \frac{5}{6}; \frac{3}{4}.$

13. $-2; 0; 4.$ 14. $\frac{3}{5}; -\frac{2}{3}; -5.$ 15. $0; 2; -2; 4.$

16. $7; 0; 1; -1.$

17. Determine the sum and product of the roots of the equation $ax^2-b^2x=ka^2$.

18. If k and l are the roots of the equation

$$5x^2-12x+2=0,$$

find the value of k^2+l^2, and of $k-l$.

19. If α and β are the roots of $ax^2+bx+c=0$, find the value of $\alpha^2+\beta^2$, and $\alpha^3+\beta^3$.

20. If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

21. If α and β be the roots of the equation

$$ax^2+bx+c=0,$$

form the equation whose roots are $\alpha^2+\beta^2$ and $-(\alpha-\beta)^2$.

22. If α and β be the roots of $ax^2+bx+c=0$, prove that the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ will be

$$x^2+\frac{2ac-b^2}{ac}x+1=0.$$

CHAPTER XVIII.

SIMULTANEOUS EQUATIONS INVOLVING THE SECOND POWER OF THE UNKNOWN LETTERS.

74. THERE are various methods for the solution of simultaneous equations of an order higher than the first; definite rules may be given for some of them, but there are others which require a combination of methods, and the student can only learn to solve them after considerable experience.

75. I. If one of the given equations is of two dimensions and can be divided by the other, the new equation thus obtained must be combined with the more simple of the original equations; the solution will then be effected as in simple equations.

Ex. 1. Solve $x^2-4y^2=33$; $x+2y=11$.

Divide the first equation by the second, and we obtain

$$x-2y=3,$$

but $$x+2y=11,$$

$\therefore$ by addition $$2x=14,$$

$$x=7, \text{ and } y=2. \quad \text{Ans.}$$

76. II. If the product of the unknown quantities be given, and also their sum or the sum of their squares, the equations must be so combined as to determine the difference of the quantities; similarly if the difference be given, the sum must be determined.

A similar process may be employed when the sum or difference of the quantities is given as one equation, and the sum of the squares as the other.

Ex. 2. Solve $9x^2+y^2=25$; $xy=4$.

The first expression may be made a perfect square by adding to it $6xy$, which from the second equation is known to be equal to 24.

Thus $$9x^2+6xy+y^2=25+24=49.$$

By taking the square root, $3x+y=\pm 7$ ..... (1).

Again, subtract $6xy$ from the first expression; then

$$9x^2-6xy+y^2=25-24=1.$$

By taking the square root, $3x-y=\pm 1$ ..... (2).

Add (1) and (2); then $6x=\pm 8$ or ± 6,

$$\left.\begin{aligned} \therefore x &= \pm\tfrac{4}{3} \text{ or } \pm 1, \\ \text{and } y &= \pm 3 \text{ or } \pm 4. \end{aligned}\right\} \text{ Ans.}$$

Ex. 3. Solve $xy=10$; $2x+3y=16$.

Our object must be to find the value of $2x-3y$; to obtain this, square the second equation; then

$$4x^2+12xy+9y^2=256,$$

from this subtract $\quad 24xy \quad =240,$

hence $\quad 4x^2-12xy+9y^2=16,$

take the square root $\quad 2x-3y=\pm 4,$

but $\quad 2x+3y=16,$

$$\therefore 4x=20 \text{ or } 12,$$

$x=5$ or 3; and $y=2$ or $\frac{10}{3}$. Ans.

77. III. When neither of the above methods is applicable, we can sometimes substitute in the less simple equation the value of one of the letters determined from the other equation.

Ex. 4. Solve $2x^2+xy+y^2=8$; $2x+3y=8$.

From the second equation we might find the value of x and substitute in the first; but as this would involve a fraction, it is more simple to multiply the first equation by 2, so that

$$4x^2+2xy+2y^2=16.$$

Now $2x=8-3y$, and substituting this value, we have

$$(8-3y)^2+(8-3y)\,y+2y^2=16,$$

$$64-48y+9y^2+8y-3y^2+2y^2=16,$$

$$8y^2-40y+48=0,$$

$$y^2-5y+6=0,$$

$$\left.\begin{array}{r} y=2 \text{ or } 3, \\ \text{and } x=1 \text{ or } -\frac{1}{2} \end{array}\right\} \text{ Ans.}$$

78. IV. A method similar to that in I. may be employed, even when one of the equations is of more than two dimensions.

Ex. 5. Solve $x^3+8y^3=35$; $x+2y=5$.

Dividing the first equation by the other, we obtain

$$x^2-2xy+4y^2=7 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Then by squaring the second equation

$$x^2+4xy+4y^2=25 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Subtract (3) from (4), and we find

$$6xy=18 \text{ or } xy=3.$$

Now, since $x+2y$ is given, our object is to determine the value of $x-2y$; to do this, we first obtain its square by subtracting $8xy$ $(=24)$ from (4); thus

$$x^2-4xy+4y^2=1,$$

$$x-2y=\pm 1,$$

but $x+2y=5,$

$\therefore 2x=6$ or 4, and $x=3$ or 2, $y=1$ or $1\frac{1}{2}$. } Ans.

Ex. 6. Solve $x+y=5$; $x^5+y^5=275$.

By dividing, we have $x^4-x^3y+x^2y^2-xy^3+y^4=55$,

also by taking the fourth power of the first equation

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=625,$$

hence, by subtracting, $5x^3y+5x^2y^2+5xy^3=570$,

$$xy(x^2+xy+y^2)=114,$$

but by squaring the first equation

$$x^2+2xy+y^2=25,$$

$$x^2+xy+y^2=25-xy;$$

substitute this in the last equation, and we obtain

$$xy(25-xy)=114.$$

Solve this as an equation with xy for the unknown quantity,

$$x^2y^2 - 25xy + 114 = 0,$$

$$(xy - 6)(xy - 19) = 0,$$

$$xy = 6 \text{ or } 19,$$

but $$x^2 + 2xy + y^2 = 25,$$

hence $$x^2 - 2xy + y^2 = 1 \text{ or } -51,$$

$$x - y = \pm 1:$$

(the other root would produce impossible values.)

Now $$x + y = 5,$$

$$\therefore 2x = 6 \text{ or } 4, \quad \therefore x = 3 \text{ or } 2, \quad y = 2 \text{ or } 3.$$ Ans.

79. V. When all the literal terms of both equations are of two dimensions, the solution is effected by writing one letter as a multiple or measure of the other ($x = vy$); then dividing one equation by the other, we shall obtain a new equation containing only the unknown multiple or measure: when its value is found, we can determine the unknown quantities from either of the given equations.

Ex. 7. Solve $2x^2 - xy + 3y^2 = 78$; $x^2 + 4xy + 2y^2 = 119$.

Let $x = vy$, and divide one equation by the other, then

$$\frac{2v^2y^2 - vy^2 + 3y^2}{v^2y^2 + 4vy^2 + 2y^2} = \frac{78}{119}.$$

Remove y^2 from numerator and denominator, and the equation becomes

$$160v^2 - 431v = -201,$$

whence $$v = \frac{3}{5} \text{ or } \frac{67}{32},$$

$$\therefore x = \frac{3y}{5} \text{ or } \frac{67y}{32}:$$

substituting this value of x in the first equation, we obtain

$$y^2 = 25 \text{ or } \frac{1024}{127},$$

$$\left.\begin{aligned} \therefore\ y &= \pm 5 \text{ or } \pm \frac{32}{\sqrt{127}}, \\ \text{and } x &= \pm 3 \text{ or } \pm \frac{67}{\sqrt{127}}. \end{aligned}\right\} \text{ Ans.}$$

80. The above are the most common methods of solution of equations of this kind, but the work may often be shortened in various other ways, such as resolving the equations into factors, or combining them by addition or subtraction either as they stand or after multiplying one or both of them, and so on. We need not do more than offer these suggestions; the student will learn from experience when they are to be applied.

Exercise XVIII.

Solve the following Simultaneous Equations:

1. $x^2 - y^2 = 80$; $x - y = 8$.
2. $4x^2 - 9y^2 = 7$; $2x + 3y = 7$.
3. $x^2 + xy = 6$; $x + y = 3$.
4. $x^2 - xy = 25$; $x - y = 5$.
5. $3x - 5y = 10$; $9x^2 - 25y^2 = 200$.
6. $4(x + 2y) = 12$; $x^2 - 4y^2 = 33$.
7. $8x + 18y = 42$; $16x^2 - 81y^2 = 63$.
8. $49x^2 - 9y^2 = 187$; $21x - 9y = 51$.
9. $x^2 + y^2 = 61$; $xy = 30$.
10. $x + y = 17$; $xy = 72$.
11. $xy = 27$; $x - y = 6$.
12. $xy = 8$; $3x - y = 10$.
13. $4x + y = 13$; $xy = 3$.
14. $5x - 2y = 26$; $xy = 12$.

15. $7x-10y=18$; $xy=4$.
16. $x^2+y^2=25$; $x+y=7$.
17. $x+y=5$; $x^2+y^2=13$.
18. $x-y=2$; $x^2+y^2=34$.
19. $x^2+y^2=50$; $x-y=6$.
20. $x-2y=2$; $x^2+4y^2=100$.
21. $x-3y=1$; $x^2-2xy+9y^2=17$.
22. $x^2+xy+4y^2=64$; $x-2y=2$.
23. $x+y=4$; $2x^2+3y^2=21$.
24. $x+2y=4$; $3x^2+y^2=13$.
25. $3x-y=5$; $x^2-2xy+3y^2=3$.
26. $2x+3y=4$; $6x^2+3xy=120$.
27. $5x+y=3$; $2x^2-3xy-y^2=1$.
28. $2x^2-5xy+3y^2=0$; $5y-2x=4$.
29. $x^2-2y^2=71$; $x+y=20$.
30. $7x^2-8xy=159$; $5x+2y=7$.
31. $2x+3y=12$; $7xy-4x^2=6$.
32. $x^2-x=5y^2$; $x^2=21+y^2$.
33. $\frac{x}{2}+\frac{y}{3}=9$; $\frac{x^2}{4}+\frac{y^2}{6}=49$.
34. $\frac{x}{7}-\frac{y}{12}=\frac{1}{2}$; $x^2-\frac{y^2}{6}=43$.
35. $\frac{1}{4}(x+y)=\frac{1}{3}(x-y)$; $\frac{x^2}{7}+2(x-y)^2=79$.
36. $x^3+y^3=9$; $x+y=3$.
37. $x+y=6$; $x^3+y^3=126$.
38. $x-y=3$; $x^3-y^3=117$.
39. $x^3-y^3=208$; $x-y=4$.
40. $x^3+8y^3=91$; $x+2y=7$.
41. $27x^3-y^3=19$; $3x-y=1$.

42. $8x^3-125y^3=91$; $2x-5y=1$.

43. $27x^3-8y^3=1512$; $3x-2y=6$.

44. $x^3+y^3=7(x+y)=35$.

45. $x^3+y^3=126$; $x^3-xy+y^3=21$.

46. $x^2+xy+y^2=7$; $x^3-y^3=35$.

47. $8x^3-y^3=9$; $4x^2+2xy+y^2=3$.

48. $27x^3+8y^3=91$; $9x^2-6xy+4y^2=13$.

49. $x^5+y^5=33$; $x+y=3$. 50. $x^5-y^5=242$; $x-y=2$.

51. $x^5-32y^5=211$; $x-2y=1$.

52. $243x^5+32y^5=781$; $3x+2y=1$.

53. $x-3y=1$; $x^5-243y^5=781$.

54. $x^5-y^5=\frac{211}{32}$; $x-y=\frac{1}{2}$.

55. $x^2+xy=20$; $xy+y^2=5$.

56. $x^2-xy=3$; $y^2-xy=-2$.

57. $x^2+3xy=40$; $4y^2+xy=9$.

58. $9x^2+2y^2=68$; $6xy-y^2=32$.

59. $\frac{x^2}{4}+\frac{y^2}{3}=7$; $xy-\frac{2y^2}{3}=6$.

60. $\frac{4x^2}{9}-xy=7$; $y^2-\frac{xy}{3}=2$.

61. $\frac{x}{y}+\frac{y}{x}=2\frac{1}{2}$; $x+y=6$.

62. $\frac{x^2}{y}+\frac{y^2}{x}=12$; $\frac{1}{x}+\frac{1}{y}=\frac{1}{3}$.

63. $2x+5y=4$; $12x^2+5y^2=12\frac{4}{5}$.

64. $x+4y=12$; $x^2-12y^2=97$.

65. $\left.\begin{aligned} x^2+xy&=4x-2,\\ y^2+xy&=4y-1.\end{aligned}\right\}$ 66. $x^2=4(2y-1);\ xy=2(x-1).$

67. $\left.\begin{aligned} x(x-y)&=2ax-2by-4b(a-b),\\ y(x-y)&=2ay-2bx+4b(a-b).\end{aligned}\right\}$

68. $\left.\begin{aligned} x^2+2yz&=y^2+2zx=a,\\ z^2+2xy&=c.\end{aligned}\right\}$

69. $x+y-xy=1;\ 13xy-x^2-y^2=29.$

70. $x^2+xy=3y^2-xy;\ 4x^2+5y^2=9.$

71. $4x^2-3xy=10;\ 2xy+5y^2=16.$

72. $3x^2+4xy=36;\ 2x^2+3y^2=35.$

73. $x^2-2xy=21;\ xy+y^2=18.$

74. $4x^2+7xy=58;\ 5y^2-xy=39.$

75. $x^2-5xy=14;\ 4y^2+3xy=25.$

76. $3x^2-xy+2y^2=36;\ 2x^2-3y^2+19=0.$

77. $4xy+5y^2=33;\ x^2+2xy+2y^2=13.$

78. $3x^2+2xy+y^2=4x^2-xy-3y^2=8.$

79. $x^2-xy+y^2=7,\ x^4+x^2y^2+y^4=133.$

80. $x^2-y^2=24;\ x^3-x^2y-xy^2+y^3=96.$

81. $x-y=2;\ x(2x+y)=40.$

82. $(x-y)(x-2y)=5;\ 2x+y=10.$

83. $x^2+y^2-x-y=78;\ xy+x+y=39.$

84. $x^2(x^2-1)+y^2(y^2-1)=84;\ x^2+x^2y^2+y^2=49.$

85. $y(y+1)=14-x(x-1);\ xy=4.$

86. $\left.\begin{aligned} x^2+y^2-x^2y-xy^2+2xy&=3,\\ x^2y(y+2)+xy^2(x+2)&=20.\end{aligned}\right\}$

87. $yz-a^2=x(y+z-x);\ zx-b^2=y(z+x-y);$
$xy-c^2=z(x+y-z).$

88. $x(y+z-x)=a^2,\ y(z+x-y)=b^2,\ z(x+y-z)=c^2.$

CHAPTER XIX.

Problems producing Quadratic Equations.

81. Ex. 1. Find two numbers such that the sum of their squares exceeds twice their product by 9, and the difference of their squares is 51.

Let x and y be the numbers;

then $x^2+y^2=2xy+9$ ..... (1),

$x^2-y^2=51$ ..... (2).

From (1) $x^2-2xy+y^2=9$,

$x-y=3$.

Divide (2) by this;

$x+y=17$,

$\therefore\ x=10;\quad y=7.$ Ans.

82. Ex. 2. A furniture dealer bought 10 tables and 30 chairs for £67. 10*s.* at such prices that for £15 he got 7 more chairs than he did tables: find the price of each.

Let x be the price of a table in £s,

y ..... chair .....

then $10x+30y=67\frac{1}{2}$ .....(1).

Now for £15 he would have got $\frac{15}{x}$ tables, or $\frac{15}{y}$ chairs,

hence $\frac{15}{x}=\frac{15}{y}-7$.....(2).

From (1) $20x=135-60y$,

$4x=27-12y$ .....(3).

Multiply together (2) and (3),

$$60=(27-12y)\left(\frac{15}{y}-7\right),$$

$$60=\frac{405}{y}-180-189+84y,$$

$$84y^2-429y+405=0,$$

$$28y^2-143y+135=0,$$

$$(4y-5)(7y-27)=0,$$

$$y=\frac{5}{4} \text{ or } \frac{27}{7},$$

the latter value would make x negative, the former value gives

$$x=8.$$

Each table cost £3; each chair £1. 5*s*. Ans.

83. Ex. 3. Find the price per dozen at which eggs are selling when a reduction of 1*d*. per dozen will give 5 more for half-a-crown.

Let x pence be the price of 12 eggs,

then for 30 pence we should buy $\frac{30\times 12}{x}\left(=\frac{360}{x}\right)$ eggs;

when the price is $x-1$ pence for 12 eggs, we should get

$$\frac{360}{x-1} \text{ eggs for 30 pence:}$$

hence

$$\frac{360}{x-1}=\frac{360}{x}+5;$$

$$\therefore\ 360x=360x-360+5x^2-5x,$$

$$5x^2-5x-360=0,$$

$$x^2-x-72=0,$$

$$x=9, \text{ or } -8.$$

Hence the price is 9*d*. per dozen. Ans.

84. Ex. 4. Determine the percentage which is being charged, when the difference between the amount at simple interest for one year and the present value for one year is to the principal in the proportion of 33 : 272.

Let P represent the number of £s in the principal, and x the rate per cent.;

then the amount at the end of the year is

$$P+\frac{Px}{100}$$

$$=P.\frac{100+x}{100},$$

also the present value for one year

$$=\frac{100\,P}{100+x},$$

hence we have the equation

$$P.\frac{100+x}{100}-\frac{100\,P}{100+x} : P :: 33 : 272,$$

$$\frac{(100+x)^2-100^2}{100\,(100+x)}=\frac{33}{272},$$

$$\frac{200x+x^2}{25\,(100+x)}=\frac{33}{68},$$

$$68x^2+12775x-82500=0,$$

$$(4x-25)\,(17x+3300)=0;$$

$$\therefore\ 4x=25,\quad \text{or } x=6\tfrac{1}{4}.\quad \text{Ans.}$$

Exercise XIX.

1. Find a number such that the double of its square increased by 21 is equal to seventeen times the original number.

2. Find two numbers whose sum is 72, and the sum of their squares 3744.

3. There is a certain number, the cube root of which is one-third of the square root: find it.

4. The difference of two numbers is three-eighths of the greater, and the sum of their squares is 356: find them.

5. The sum of two numbers is eight times their difference, and the difference of their squares 128: find them.

6. Find two numbers such that their product is 24, and the sum of their squares 577.

7. Find two numbers such that their product divided by their sum gives 3 as quotient and 3 as remainder, while their sum divided by their difference gives 3 as quotient and 2 as remainder.

8. There are two numbers such that the square of the greater divided by the square of the less gives 5 as quotient, and 4 as remainder; while the square of the less divided by the greater gives 1 as quotient and 2 as remainder: find the numbers.

9. The product of two consecutive numbers is less than double the square of the greater by 132: find them.

10. Find the number whose square added to its cube is nine times the next higher number.

11. What is the number consisting of two digits whose product is 12, such that if the order of the digits be reversed, the number is increased by 36?

12. The two digits of a certain number differ by 1, and their product is one-fourth of the next higher number: find the number.

13. Determine the length of the side of a square such that the number of square feet in its area exceeds the number of linear feet in its perimeter by 12.

14. The perimeter of a rectangle is 34 feet; if its length were increased by 4 feet, while its perimeter remained unchanged its former area would exceed the double of the new area by 6 square feet: find the original length.

15. If 3 yards be taken from one side of a rectangle whose perimeter is 14 yards, and added to the other side, its area is doubled: find the lengths of the sides.

16. 2120 soldiers are drawn up on a square piece of ground; the men in front are 4 feet apart and the ranks are a yard apart: how many ranks are there?

17. The forewheel of a carriage makes 6 revolutions more than the hind-wheel in 120 yards, and the circumference of one is a yard less than the other: find that of each.

18. *A* and *B* started on velocipedes at the same time, *A* to go from Windsor to London, and *B* from London to Windsor. *A* reached London $2\frac{1}{4}$ hours, and *B* reached Windsor 1 hour, after they met. In what time did each perform the journey?

19. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is one mile an hour less than his rate during the first half, and he reaches the summit in $4\frac{3}{8}$ hours. He descends in $3\frac{1}{2}$ hours by walking at an uniform rate, which is one mile an hour more than that during the first half of his ascent. Find the distance to the summit and his rate of walking at starting.

20. A water-butt receives the rain water from two roofs, and during a steady rain it is observed that the rain from one roof alone would fill the butt in 4 hours less than that from the other: it is filled by both together in $3\frac{3}{4}$ hours. In what time would the water from each alone fill the butt?

21. In laying down a croquet-ground, I receive bills for 252 square yards of turf, and for 64 yards of wire-netting with which it is enclosed. What are the dimensions of the sides of the ground?

22. A man bought 7 photographs and 12 prints for £50; he bought 3 more photographs for £10 than he did prints for £6: what was the price of a photograph?

23. The price of four first class and five third class tickets between London and Edinburgh is £22: also for £56 the Company will sell 19 more third class than first class tickets: determine the price of a first class ticket.

24. A man sold 5 sheep and 8 calves for £30; he sold 2 more sheep than he sold calves for £20: what did he receive for each?

25. What is the present price of wine per dozen when since the price was lowered 10 shillings per dozen, we get 6 bottles more for £5?

26. How many apples are sold for 1*s.* when by giving 4 more for 6*d.* the price is reduced by 3*d.* a dozen?

27. At what price are eggs selling when an increase in price of 2*d.* a dozen will give one less in a shilling's worth?

28. Find the price of meat when if the price were raised 10*d.* per stone of 8 lbs. you would get $2\frac{1}{2}$ lbs. less for a sovereign.

29. A man divided his property £9207 into two parts in the ratio of 176 to 165, one of which he placed out at compound interest and the other at the same percentage at simple interest; at the end of two years he found that the difference in their amounts was £1023: what was the rate of interest?

30. A man invests his money at a certain rate of interest for two years, and finds that he will get 1 per cent. more for it if he reckon by compound interest than he will if he reckon by simple interest: what is the rate of interest?

31. What is the percentage at which money is borrowed when the difference between the amount at simple interest for one year and the present value for one year is to the principal as 51 : 650?

32. A person divided his money into two portions, which when put out at interest at rates differing by 1 per cent. produced equal amounts of interests. If the first portion had been put out at the second rate of interest it would have produced £25; and if the second portion had been put out at the first rate of interest it would have produced £16: what were the rates of interest?

33. Find a number (less than 200) which is formed by adding all the products, taken two at a time, of three consecutive even numbers, it being given that of the three digits which compose the required number the last two are the same as the middle even number.

34. A and B are towns situated 18 miles apart on the same bank of a river. A man goes from A to B in 4 hours by sculling the first half of the distance and walking the second half. In returning he walks the first half at the

same rate as before, but the stream being with him he sculls $1\frac{1}{2}$ miles per hour more than in going, and accomplishes the whole distance in $3\frac{1}{2}$ hours. Find his rate of walking.

35. A merchant having bought a quantity of wheat, found that five-ninths of it was damaged. By selling the damaged and the undamaged portions at different rates of profit he made the same by each of them. His average profit being 5 per cent., what profit did he make by the damaged portion?

36. A person bought a certain number of shares, when they were at a discount, for £9900; afterwards when they rose to a premium of the same rate per cent. he sold just enough stock to save him from any possible loss, and he then had 20 shares left; what was the price he paid for them?

CHAPTER XX.

CUBE ROOT.

85. THE method of finding the cube root of an algebraical expression will be most easily learnt from the explanation of simple examples.

Ex. 1. Extract the cube root of $8x^3+36x^2+54x+27$.

$8x^3+36x^2+54x+27 \,(2x+3$ Ans.

$8x^3$

$6x+3$	$12x^2+18x+9$	$36x^2+54x+27$
		$36x^2+54x+27$

The cube root of the first term ($8x^3$) is determined by inspection, and placed in the quotient. We then arrange the work in columns, regarding the given expression as the head of a third column; in the first column we write *the quotient multiplied by* 3, ($2x \times 3 = 6x$); in the second column (which corresponds to the column containing the divisor in square root) we write *the square of the quotient multiplied by* 3, $\{(2x)^2 \times 3 = 12x^2\}$. To determine the next term of the quotient we divide the dividend by the term in the second column, which gives 3: this is used in *four* ways: (1) it is added to the quotient, (2) it is added to the first column, (3) the first column thus completed is multiplied by it, and the result added to the second column, (4) the second column thus completed is multiplied by it and placed under the dividend.

86. **Ex. 2.** Extract the cube root of

$$64x^6 - 384x^5 + 864x^4 - 896x^3 + 432x^2 - 96x + 8.$$

$64x^6 - 384x^5 + 864x^4 - 896x^3 + 432x^2 - 96x + 8$ $(4x^2 - 8x + 2$ Ans.

$64x^6$

$12x^2 - 8x$	$48x^4 - 96x^3 + 64x^2$	$-384x^5 + 864x^4 - 896x^3$ $-384x^5 + 768x^4 - 512x^3$
$12x^2 - 24x + 2$	$48x^4 - 192x^3 + 192x^2$ $24x^2 - 48x + 4$	$96x^4 - 384x^3 + 432x^2 - 96x + 8$ $96x^4 - 384x^3 + 432x^2 - 96x + 8$
	$48x^4 - 192x^3 + 216x^2 - 48x + 4$	

The method in the first part of this is identical with that in Ex. 1. When we have obtained the second remainder, the same principle is extended to the two terms of the quotient: in the first column we write $12x^2 - 24x$ (i.e. $(4x^2 - 8x) \times 3$); for the second column we take the square of the quotient and multiply it by 3, $\{(4x^2 - 8x)^2 \times 3 = 48x^4 - 192x^3 + 192x^2\}$. Dividing this into the dividend we obtain the quotient 2, which as before is used in four ways.

87. The sixth root of any expression is obtained by first finding its square root, and then the cube root of this square root.

88. When the cube root of a number is to be determined, the figures composing it are marked off in batches of three, beginning at the right hand; the cube root is then found on the same principle as for an algebraical expression.

Ex. 3. Extract the cube root of 438976.

		438'976 (76 Ans.
		343
210+6	14700+1296	95976
=216	=15996	95976

We mark off the last 3 figures, and proceed by trial to find the nearest cube number below 438; this is found to be 7^3 (=343). For the dividend we have the remainder 95 followed by the other three figures 976. In forming the first and second columns, it must be carefully remembered that the figure 7 in the quotient stands for 70; hence the first column is 70×3 (=210), and the second column $70^2 \times 3$ (=14700). Dividing the second column into the dividend we obtain 6, which is placed in the quotient and added to the first column: 216 is then multiplied by 6 and the result added to the second; lastly, the completed second column is multiplied by 6, and placed under the dividend.

89. Ex. 4. Extract the cube root of 32157432.

		32'157'432 (318 Ans.
		27
90+1	2700+91	5157
=91	=2791	2791
930+8	288300+7504	2366432
=938	=295804	2366432

It will be observed that in forming the *first* divisor the quotient 3 is regarded as standing for 30 (not 300), because the last three figures of the question are not then under consideration, and the figure 3 is in the place which has 10 times (not 100 times) the value of the figure which is being determined.

90. When a part or the whole of the given number is a decimal fraction, the same principles apply as in the extraction of the square root: the first three figures after the decimal point must be taken together, and if less than three be given, they must be completed by placing after them 0 or 00, according as we have in the given number two figures or one.

Exercise XX.

Find the cube root of

1. $a^3+3a^2x+3ax^2+x^3$.
2. $8x^3-12x^2+6x-1$.
3. $27a^6-54a^4x+36a^2x^2-8x^3$.
4. $64x^3+48x^2y^2+12xy^4+y^6$.
5. $125a^9-150a^6b+60a^3b^2-8b^3$.
6. $1728x^{12}-4320x^9y^3+3600x^6y^6-1000x^3y^9$.
7. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
8. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$.
9. $8a^{12}-36a^{10}+102a^8-171a^6+204a^4-144a^2+64$.
10. $8-48ax^2+108a^2x^4-112a^3x^6+54a^4x^8-12a^5x^{10}+a^6x^{12}$.
11. $27x^6-27x^5y-45x^4y^2+35x^3y^3+30x^2y^4-12xy^5-8y^6$.
12. $64a^{12}+240a^{10}b+156a^8b^2-235a^6b^3-117a^4b^4+135a^2b^5-27b^6$.

13. $8a^3-12a^2+30a+23+30a^{-1}-12a^{-2}+8a^{-3}$.

14. $x^{\frac{3}{4}}+6x^{\frac{5}{8}}y^{-1}+3x^{\frac{1}{2}}y^{-2}-28x^{\frac{3}{8}}y^{-3}-9x^{\frac{1}{4}}y^{-4}+54x^{\frac{1}{8}}y^{-5}-27y^{-6}$.

15. $a^2-3a^{\frac{5}{3}}b^{-1}-3a^{\frac{4}{3}}b^{-2}+11ab^{-3}+6a^{\frac{2}{3}}b^{-4}-12a^{\frac{1}{3}}b^{-5}-8b^{-6}$.

16. $27x^3-54x^{\frac{5}{2}}y^{\frac{1}{8}}+63x^2y^{\frac{1}{4}}-44x^{\frac{3}{2}}y^{\frac{3}{8}}+21xy^{\frac{1}{2}}-6x^{\frac{1}{2}}y^{\frac{5}{8}}+y^{\frac{3}{4}}$.

Find the sixth root of

17. $a^{12}-12a^{10}x^2+30a^8x^4-160a^6x^6+240a^4x^8-192a^2x^{10}+64x^{12}$.

18. $729a^6+1458a^5x+1215a^4x^2+540a^3x^3+135a^2x^4+18ax^5+x^6$.

19. $729a^{12}+729a^{10}x+\frac{1215}{4}a^8x^2+\frac{135}{2}a^6x^3+\frac{135}{16}a^4x^4+\frac{9}{16}a^2x^5+\frac{1}{64}x^6$.

Find the ninth root of

20. $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2y^7+9xy^8+y^9$.

Find the cube root of

21.	13824.	22.	46656.	23.	85184.
24.	531441.	25.	474552.	26.	658503.
27.	804357.	28.	175616.	29.	1225043.
30.	1953125.	31.	2685619.	32.	1481544.
33.	5735·339.	34.	14706125.	35.	22·425768.
36.	28·372625.	37.	23887872.	38.	45118·016.
39.	356400829.	40.	268336125000.	41.	5890514616.

42. 141176·604743.

CHAPTER XXI.

RATIO.

91. RATIO is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part, or parts one is of the other. Thus, if we compare 6 with 3 we observe that it has a certain magnitude with respect to 3, which it contains twice; again, if we compare it with 2 we see that it has a different relative magnitude, for it contains 2 three times; in other words 6 is greater when compared with 2 than it is when compared with 3.

92. The ratio of a to b is usually expressed by two points placed between them, thus $a:b$; the former, a, is called the *antecedent* of the ratio; the latter, b, is called the *consequent*,

The fraction $\frac{a}{b}$ also expresses the relation which a bears to b in respect of magnitude; hence the ratio $a:b$ may also be written as the fraction $\frac{a}{b}$, or vice versâ.

Ex. 1. What is the ratio of £1. 8*s*. 9*d*. to £10?

Reducing one quantity to the fraction of the other, we obtain $\frac{23}{160}$; hence the ratio is 23 : 160.

93. To compare the values of any set of ratios, we express them as fractions, and by reducing them to a common denominator determine the order of their value.

Ex. 2. Compare the following ratios:

$$7 : 12;\ 4 : 9;\ 11 : 18;\ 3 : 4.$$

The ratios may be written $\frac{7}{12}, \frac{4}{9}, \frac{11}{18}, \frac{3}{4}$;

and these are equal to $\frac{21}{36}, \frac{16}{36}, \frac{22}{36}, \frac{27}{36}$.

Hence their order of value is $3 : 4;\ 11 : 18;\ 7 : 12;\ 4 : 9$.

94. Ratios are *compounded* by multiplying together the fractions which are equivalent to them.

Ex. 3. Compound the ratios $7 : 15;\ 2\frac{1}{2} : 8\frac{3}{4};\ 3 : 2$.

The required result $= \frac{7}{15} \times \frac{2\frac{1}{2}}{8\frac{3}{4}} \times \frac{3}{2}$

$$= \frac{7}{15} \times \frac{10}{35} \times \frac{3}{2} = \frac{1}{5}.$$

Hence the compounded ratio is $1 : 5$. Ans.

95. Ex. 4. What number must be added to each term of the ratio $9 : 14$, that it may become equal to $3 : 4$?

Let x be the required number, then since it is to be added to both terms of the ratio, we have the equation

$$\frac{9+x}{14+x} = \frac{3}{4},$$

$$36 + 4x = 42 + 3x,$$

$$x = 6. \quad \text{Ans.}$$

Ex. 5. Two numbers are in the ratio of $3 : 5$; if 5 be taken from each they will be in the ratio of $1 : 2$; find them.

Let x and y be the required numbers,

then $$\frac{x}{y}=\frac{3}{5} \quad \ldots\ldots (1),$$

$$\frac{x-5}{y-5}=\frac{1}{2} \quad \ldots\ldots (2).$$

These equations become $\left.\begin{array}{l} 5x-3y=0, \\ 2x-\ y=5, \end{array}\right\}$

from which we obtain $x=15,\ y=25$. Ans.

96. A ratio is said to be a ratio *of greater inequality, of less inequality*, or *of equality*, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

97. *A ratio of greater inequality is diminished and a ratio of less inequality is increased by adding any quantity to both its terms.*

Let $a : b$ be the ratio, and let x be added to both its terms so that a new ratio is formed, viz. $a+x : b+x$.

Now $a : b$ is $>$ or $<$ $a+x : b+x$,

according as $\frac{a}{b}$ is $>$ or $<$ $\frac{a+x}{b+x}$,

or as $a(b+x)$ is $>$ or $<$ $b(a+x)$,

or as ax is $>$ or $<$ bx,

or as a is $>$ or $<$ b.

In the same way it may be shewn that a ratio of greater inequality is increased and a ratio of less inequality diminished by taking any quantity from both its terms.

98. The following is a very important theorem, which is frequently of use in the solution of questions in ratio.

If any number of ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$... *are equal to each other, then each of them is equal to the ratio* $\frac{la+mc+ne+\ldots}{lb+md+nf+\ldots}$, *where* l, m, n... *are any numbers whatever.*

Let each of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\ldots = k$,

then $$a=bk,\ c=dk,\ e=fk\ldots\ldots,$$

hence we have

$$\frac{la+mc+ne+\ldots}{lb+md+nf+\ldots} = \frac{lbk+mdk+nfk+\ldots}{lb+md+nf+\ldots}$$

$$= k,$$

but k is equal to each of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\ldots$,

therefore each of these ratios is equal to

$$\frac{la+mc+ne+\ldots}{lb+md+nf+\ldots}.$$

In a similar manner it may be shewn by first raising the ratios to the pth power that each of them is equal to

$$\left\{\frac{la^p+mc^p+ne^p+\ldots}{lb^p+md^p+nf^p+\ldots}\right\}^{\frac{1}{p}}.$$

99. The student should note the above method of assuming some value (like k) for each of the ratios; he will find that it may be applied in many cases where the theorem itself is not required.

Ex. 6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

$$\sqrt[3]{\frac{a^6+la^2c^4+me^6}{b^6+lb^2d^4+mf^6}}=\frac{2a^2-c^2}{2b^2-d^2}.$$

Let $$k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$

then $$a=bk, \quad c=dk, \quad e=fk;$$

$$\therefore \sqrt[3]{\frac{a^6+la^2c^4+me^6}{b^6+lb^2d^4+mf^6}}=\sqrt[3]{\frac{b^6k^6+lb^2d^4k^6+mf^6k^6}{b^6+lb^2d^4+mf^6}}=\sqrt[3]{k^6}=k^2,$$

also $$\frac{2a^2-c^2}{2b^2-d^2}=\frac{2b^2k^2-d^2k^2}{2b^2-d^2}=k^2.$$

Hence the two given expressions are equal to each other.

Exercise XXI.

Compare the values of the following sets of ratios:

1. 2 : 3, 4 : 5, 9 : 10, 13 : 15.
2. 11 : 12, 7 : 10, 17 : 20, 13 : 18.
3. 2 : 5, 61 : 70, 5 : 6, 4 : 7.
4. $a+b : a-b,\ 2ab : a^2-b^2$.

Compound the following ratios:

5. 6 : 7, 5 : 8, 4 : 9.
6. 11 : 20, 9 : 14, 2 : $5\frac{1}{2}$, $3\frac{1}{2}$: 6.
7. $a+b : a-b,\ a^2-b^2 : a^2-4b^2,\ ab+2b^2 : (a+b)^2$.
8. $2x : 3y,\ x+y : 4x,\ x^2-xy : xy+y^2$.
9. What is the ratio of £2. 10*s.* to £8. 2*s.* 6*d.*?

10. Find the ratio of the rates of two trains, one of which goes $42\frac{3}{4}$ miles an hour, and the other 99 miles in $2\frac{3}{4}$ hours.

11. What number must be added to each term of the ratio 3 : 5 that it may become equal to the ratio 5 : 6?

12. What number must be taken from each term of the ratio 19 : 30 that it may become 1 : 2?

13. What number must be taken from the second term of the ratio 21 : 34 and added to the first term that it may equal 5 : 6?

14. Two numbers are in the ratio of 3 : 4; if 8 be added to each they will be in the ratio of 5 : 6; find the numbers.

15. Find two numbers in the ratio of 8 : 9, such that if 2 be added to each they are in the ratio of 9 : 10.

16. Two numbers are in the ratio of 2 : 3; if 3 be added to each they will be as 5 : 7; find them.

17. What number must be added to each term of the ratio 8 : 13 that it may equal the ratio of 3 : 4?

18. Two numbers are in the ratio of 3 : 4; if one be taken from the less and added to the greater they will be as 2 : 3; find them.

19. Find two numbers in the ratio of 5 : 8, such that their sum has to the difference of their squares the ratio of 1 : 12.

20. Find two numbers in the ratio of 4 : 5, such that their difference has to the sum of their squares the ratio of 1 : 82.

21. The difference of two numbers is twelve; also their sum has to the difference between the greater and three times the less the ratio of 7 : 3; find them.

22. Find two numbers whose difference is 4, such that the sum of their squares is to the square of their sum as 61 : 121.

23. Two numbers are in the ratio of 1 : 2; if 2 be added to each, the new ratio will be the duplicate of 3 : 4; find the numbers.

24. Find a number such that the ratio it bears to 18 is the duplicate of the ratio which the same number diminished by 2 bears to 9.

25. Find a number such that the ratio it bears to 8 is twice the duplicate of the ratio which the next higher number bears to 10.

26. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that each of these ratios is equal to

$$\frac{16a+4c+e}{16b+4d+f};$$

also shew that

$$\frac{3a^2+2ac+e^2}{3b^2+2bd+f^2}=\sqrt{\frac{a^4+5e^4}{b^4+5f^4}}.$$

27. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, prove that each of these fractions is equal to

$$\sqrt[4]{\frac{pa^4+qc^2e^2+raceg}{pb^4+qd^2f^2+rbdfh}}.$$

28. If $\frac{b-c}{a}=\frac{c-a}{b}=\frac{a-b}{c}$, and a, b, c are all positive, shew that $a=b=c$.

29. If $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$, and $x+y+z=a+b+c$, and l, m, n have the same signs, prove that $x=a$, $y=b$, $z=c$.

30. If $\frac{a^2-2b^2+c^2}{x^2}=\frac{b^2-2c^2+a^2}{y^2}=\frac{c^2-2a^2+b^2}{z^2}$, shew that $a^2=b^2=c^2$.

CHAPTER XXII.

PROPORTION.

100. FOUR quantities are said to be *proportionals* when the first is the same multiple, part, or parts of the second that the third is of the fourth; that is, the four quantities a, b, c, d, are called proportionals when $\frac{a}{b}=\frac{c}{d}$. This is usually expressed by saying a is to b as c is to d, and is written $a : b :: c : d$ or $a : b=c : d$.

The terms a and d are called the *extremes*; b and c the *means*.

101. *When four quantities are proportional, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the four quantities; then since they are proportional $\frac{a}{b}=\frac{c}{d}$; and by multiplying both sides of the equation by bd, we have $ad=bc$.

By the help of this, when any three terms of a proportion are given, we can determine the remaining term.

Ex. 1. Determine the value of x when $8 : 15 :: x : 20$.

Taking the products of the means and extremes, we have

$$15x = 160,$$

$$x = 10\tfrac{2}{3}. \quad \text{Ans.}$$

102. Three quantities are said to be in *continued* proportion, when the first has to the second the same ratio as the second has to the third.

When three quantities are in continued proportion, the product of the first and third is equal to the square of the second.

Let a, b, c be the three quantities, then $\frac{a}{b} = \frac{b}{c}$, and multiplying both sides by bc, we have $ac = b^2$.

In this case b is said to be a *mean proportional* between a and c.

Ex. 2. If a, b, c are in continued proportion, shew that

$$\sqrt{a^2+b^2} : \sqrt{b^2+c^2} :: b : c.$$

Since a, b, c are proportionals, $b^2 = ac$,

hence
$$\frac{\sqrt{a^2+b^2}}{\sqrt{b^2+c^2}} = \frac{\sqrt{a^2+ac}}{\sqrt{ac+c^2}} = \frac{\sqrt{a}}{\sqrt{c}} = \frac{\sqrt{ac}}{c} = \frac{b}{c};$$

$$\therefore \sqrt{a^2+b^2} : \sqrt{b^2+c^2} :: b : c.$$

103. *If four quantities be proportionals, they are also proportionals either when taken* inversely, *or when taken* alternately.

If $a : b :: c : d$,

then $b : a :: d : c$;

and also $a : c :: b : d$.

For $\frac{a}{b}=\frac{c}{d}$; divide unity by each of these equals,

then $\frac{b}{a}=\frac{d}{c}$, i.e. $b : a :: d : c$.

Again, $\frac{a}{b}=\frac{c}{d}$; multiply both sides by $\frac{b}{c}$, and we have

$\frac{a}{c}=\frac{b}{d}$, i.e. $a : c :: b : d$.

104. *When four quantities are proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

If $a : b :: c : d$,

then also $a+b : b :: c+d : d$.

Because $\frac{a}{b}=\frac{c}{d}$,

by adding 1 to each side,

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

$$\frac{a+b}{b}=\frac{c+d}{d},$$

that is, $a+b : b :: c+d : d$.

105. Several other relations may be shewn to exist between four quantities which are proportional; they may all be deduced from the proportion $\frac{a}{b}=\frac{c}{d}$ in a similar manner to the propositions in the last two sections.

Exercise XXII.

Determine the values of x which will satisfy the following proportions:

1. $6 : 27 :: 20 : x$.
2. $10 : 21 :: 25 : x$.
3. $5 : 12 :: x : 42$.
4. $8 : x :: 12 : 8$.
5. $x : 14 :: 26 : 91$.
6. $a : x :: b^2 : a^2$.
7. $5-2x : 3-x :: x+5 : 2x+3$.
8. $a+x : c-x :: a : b$.
9. If 3, x, 867 are in continued proportion, find x.
10. Find the value of x when $x-3$, $x-2$, and x are in continued proportion.

If $a : b :: b : c$, prove the following relations:

11. $a^3+c^3 : a^2-b^2+c^2 :: a+c : 1$.
12. $a^2+b^2 : b^2+c^2 :: a : c$.
13. $a^2+ab+b^2 : b^2+bc+c^2 :: a : c$.
14. $a+b : b+c :: \sqrt{a} : \sqrt{c}$.
15. Three numbers are in continued proportion; the first of them is 8, and the sum of the other two is 30; find the numbers.

If $a : b :: c : d$, prove that

16. $a : b :: \sqrt{a^2+c^2} : \sqrt{b^2+d^2}$.
17. $a^2+3ab+b^2 : c^2+3cd+d^2 :: 2ab+3b^2 : 2ad+3d^2$.
18. $3a^2+4b^2 : 6a^2-5b^2 :: 3c^2+4d^2 : 6c^2-5d^2$.
19. $(a+c)^2 : (b+d)^2 :: 2a^2+3c^2 : 2b^2+3d^2$.

20. $\sqrt[3]{a^3+pb^3} : \sqrt[3]{c^3+pd^3} :: a : c.$

21. $\sqrt{a^2+b^2} : \sqrt{c^2+d^2} :: \sqrt[3]{a^3+b^3} : \sqrt[3]{c^3+d^3}.$

22. $\dfrac{(a-b)(7a-3b)}{2a^2+5b^2} = \dfrac{7c^2-10cd+3d^2}{2c^2+5d^2}.$

23. The second and third terms of a proportion are 12 and 15; and the sum of the other two terms is 29: determine the proportion.

24. What quantity must be added to each term of the ratios 7 : 9 and 25 : 31 that the numbers may be proportional?

25. Given that $x+5 : y+2 :: 5 : 4$; and that $x^2+y^2 : x^2+2y^2 :: 25 : 41$, find x and y.

CHAPTER XXIII.

VARIATION.

106. ONE quantity is said to *vary directly* as another when they depend wholly upon each other, and in such a manner that if one be changed the other is changed in *the same proportion.* Thus, if A and B be two quantities dependent upon each other in such a way that when A is changed into a, B must be changed into another value b, such that $A : a :: B : b$, then A is said to *vary directly* as B.

The word 'directly' is frequently omitted, and one quantity is briefly said to *vary* as the other.

The symbol $\propto$ is used to express variation: thus $x \propto y$ is the brief way of writing "x varies directly as y," or, "x varies as y."

107. *If* A *varies as* B, A *is equal to* B *multiplied by some constant quantity.*

Let b be the value which B assumes when A is changed into; a then $A : a :: B : b$,

or $$A = \frac{a}{b} B.$$

Therefore A is equal to B multiplied by the constant quantity $\frac{a}{b}$.

In the solution of questions we may assume any letter for this constant multiplier, and in this way the relation between the two quantities is reduced to a simple equation.

Ex. 1. If $x \propto y$, and $x=4$ when $y=6$; find the relation between x and y.

Let $$x = ky;$$

now $$x=4, \text{ if } y=6;$$

$$\therefore\ 4=6k, \text{ or } k=\frac{2}{3}.$$

Substitute this in the above equation, and we have

$$x=\frac{2y}{3}. \quad \text{Ans.}$$

Ex. 2. If $x \propto y$, and $x=12$ when $y=15$; find y when $x=20$.

Let $$x = ky,$$

then $$12=15k, \quad \text{or } k=\frac{12}{15}=\frac{4}{5};$$

$$\therefore x=\frac{4y}{5}.$$

Hence when $x=20$, $y=\frac{5}{4}\times 20=25$. Ans.

108. One quantity is said to *vary inversely* as another when one is changed in the same proportion as the reciprocal of the other; thus if A vary inversely as B, $A\propto\frac{1}{B}$ or $A=\frac{k}{B}$.

109. One quantity is said to vary as *two others jointly*, if when the former is changed in any manner, the *product of the other two* be changed in the same proportion: thus if A vary as B and C jointly, we write $A\propto BC$ or $A=kBC$.

Ex. 3. If A vary as B and C jointly, and $A=10$ when $B=6$ and $C=1$; find A when $B=2$ and $C=9$.

Let $A=kBC$,

then $10=6k$;

$$\therefore k=\frac{5}{3}, \text{ and } A=\frac{5BC}{3}.$$

Hence when $B=2$ and $C=9$, $A=\frac{5}{3}\times 2\times 9=30$. Ans.

Ex. 4. A varies directly as B, and inversely as C; also $A=12$ when $B=4$ and $C=3$; find A in terms of B and C.

Let $A=\frac{kB}{C}$,

then $12=\frac{4k}{3}$, or $k=9$;

$$\therefore A=\frac{9B}{C}. \text{ Ans.}$$

110. *If three quantities* A, B, C, *be so connected that* A *varies as* B *when* C *is constant, and* A *varies as* C *when* B *is constant; then when* B *and* C *are both variable,* A *varies as* BC.

The variation of A depends upon the variations of B and C; let the changes take place separately, and when B is changed into b, let A be changed into a; then by the supposition

$$A : a :: B : b,$$

but this value a will again be changed by the variation of C; let then a become a' when C becomes c, that is

$$a : a' :: C : c.$$

Now combine this with the other proportion, and we have

$$Aa : aa' :: BC : bc,$$

or

$$A : a' :: BC : bc,$$

that is

$$A \propto BC.$$

111. The above theorem is readily exemplified in the solution of questions in Double Rule of Three in Arithmetic.

Ex 5. If 7 men can reap 126 acres in 12 days, how many acres can 16 men reap in 3 days?

The acres will vary as the number of men when the days are invariable, and as the number of days when the men are invariable: therefore when both days and men are variable, the acres will vary as their product.

Let A be the acres, D the days, M the men;

then $A \propto MD$, or $A = kMD$.

Now $A = 126$ when $M = 7$ and $D = 12$;

$$\therefore\ 126 = 84k, \text{ or } k = \frac{126}{84} = \frac{3}{2};$$

$$\therefore\ A = \frac{3MD}{2}.$$

Hence when $M = 16$, and $D = 3$,

$$A = \frac{3 \times 16 \times 3}{2} = 72. \quad \text{Ans.}$$

Exercise XXIII.

1. A varies as B, and when $B = 10$, $A = 16$; find A in terms of B.

2. A varies as B; when $A = 6$, $B = 10$; find A when $B = 25$.

3. A varies as B, and when $A = 2$, $B = \cdot 5$; find A in terms of B.

4. A varies as B, and when $A = 8$, $B = 5\frac{1}{3}$; find the ratio $A : B$.

5. A varies as B and C jointly; also $A = 4$ when $B = 8$, and $C = 1$; find the value of A when $B = 6$, $C = 3$.

6. A varies as B and C jointly, and when $B = 2$, $C = 5$, and $A = 7$; find the value of B when $C = 2$ and $A = 49$.

7. A varies as B and C jointly; 10, 6, and 5 are one set of values of A, B, and C respectively; find the value of B when $A = 12$, and $C = 4$.

8. A varies directly as B and inversely as C; when $B = 6$ and $C = 5$, $A = 12$; find the value of B when $A = 1\frac{1}{2}$ and $C = 40$.

9. A varies directly as B^2 and inversely as C^3; if $A=8$ when $B=2$, and $C=1$, find the value of C when $A=\frac{2}{3}$ and $B=3$.

10. If A^2 vary directly as B and inversely as C^3; and $A=2a$ when $B=a$, and $C=c^2$; find A when $B=4ac$, and $C=c$.

11. If x varies as $y+z$, and if 1, 2, 3 be simultaneous values of x, y, and z; find the value of z when $x=1$ and $y=4$.

12. If x vary as the sum of two quantities, one of which is constant and the other varies as y; and if $x=3$, when $y=1$; and $x=9$ when $y=4$; find x in terms of y.

13. If x vary as the sum of 3 quantities, one of which is constant, and the others vary as y and y^2 respectively; and if the values of x be 2, 6, 18 when $y=1$, -1, 5; find x in terms of y.

14. If in the last question the values of x be 12, 9, 60 corresponding to -2, 1, 4 for y; find x in terms of y.

15. If a vary as the sum of two quantities, one of which varies as x and the other as x^3; and if $a=7$ and 39 when $x=1$ and 3 respectively; find a in terms of x.

16. A varies as the difference of two quantities, one of which is constant and the other varies as B^2; if 3 and 18 be the values of A corresponding to 2 and 1 for B; find A in terms of B.

17. If $$x \propto \frac{u}{y^2},\ u \propto \frac{y^3}{z},\ z^2 \propto y;$$

shew that $$x \propto \sqrt{y}.$$

18. How many acres can 16 men reap in 3 days, if 7 men can reap 126 acres in 12 days?

19. If 12 horses eat 48 bushels of oats in 20 days, how many bushels will 30 horses eat in 16 days?

20. If 5 men can reap 20 acres in 2 days, how many men will be required to reap 114 acres in 3 days?

21 If the carriage of 25 cwt. for 28 miles cost £1. 14*s.* 5*d.*, what distance will 40 cwt. be carried for £2. 19*s.*?

22. In how many days will 18 men reap 114 acres, if 24 men reap 76 acres in 6 days?

23. The wages of 5 men for 7 weeks being £21. 17*s.* 6*d.*, how many men can be hired to work 4 weeks for £30?

24. The pay and provisions of 750 soldiers for 12 days amount to £900; for how many days will £3000 keep 2000 soldiers at the same rate?

CHAPTER XXIV.

THEORY OF INDICES.

112. *To prove that* $x^m \times x^n = x^{m+n}$, *when* m *and* n *are positive integers.*

It has been explained that the number of times any letter is multiplied into itself is expressed by its *index*, so that

$$x^m = x \times x \times x \ldots \text{ to } m \text{ factors},$$
$$x^n = x \times x \times x \ldots \text{ to } n \text{ factors};$$

hence

$$x^m \times x^n = (x \times x \times x \ldots \text{ to } m \text{ factors}) \times (x \times x \times x \ldots \text{to } n \text{ factors})$$
$$= x \times x \times x \ldots \text{ to } (m+n) \text{ factors}$$
$$= x^{m+n}.$$

In the same way it may be shewn that

$$x^m \times x^n \times x^p \times \ldots = x^{m+n+p+\ldots}.$$

113. *To prove that* $\frac{x^m}{x^n} = x^{m-n}$ *where* m *and* n *are positive integers, and* m *is greater than* n.

As before $x^m = x \times x \times x \ldots$ to m factors,

$$x^n = x \times x \times x \ldots \text{ to } n \text{ factors};$$

hence
$$\frac{x^m}{x^n} = \frac{x \times x \times x \ldots \text{ to } m \text{ factors}}{x \times x \times x \ldots \text{ to } n \text{ factors}}$$
$$= x \times x \times x \ldots \text{ to } (m-n) \text{ factors}$$
(since the n factors of the denominator will cancel n of those in the numerator)
$$= x^{m-n}.$$

114. *To prove that* $(x^m)^n = x^{mn}$ *when* m *and* n *are positive integers.*

$$(x^m)^n = x^m \times x^m \times x^m \ldots \text{ to } n \text{ factors}$$
$$= x^{m+m+m+\ldots \text{ to } n \text{ terms}} \text{ (by Art. 112)}$$
$$= x^{mn}.$$

115. The propositions of the last three Articles are all restricted by the condition that m and n are positive integers. It may however be asked what meanings are to be attached to the indices when they are fractional or

negative. Since we have not bound ourselves to any interpretation, we are at liberty to determine the principle in accordance with which they shall be used; it is then agreed that we attach to them a meaning such that they shall still fulfil the relation (which we have seen to be true for positive integral indices)

$$x^m \times x^n = x^{m+n}.$$

116. From this we may first determine the meaning of $x^{\frac{1}{2}}$; for let $m = n = \frac{1}{2}$; then

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x;$$

but we know that $\sqrt{x} \times \sqrt{x} = x$;

hence we see that $x^{\frac{1}{2}}$ is the same as $\sqrt{x}$.

We may extend this to determine the value of $x^{\frac{1}{n}}$; for

$$(x^{\frac{1}{n}})^n = x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times \dots \text{ to } n \text{ terms} = x^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\dots \text{ to n terms}}$$

$$= x^{\frac{n}{n}} = x;$$

$$\therefore \quad x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Similarly it may be shewn that

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}.$$

We see therefore that a fractional index denotes a root of the letter to which it is attached, the denominator of the fraction expressing what root it is.

Exs. $a^{\frac{1}{6}} = \sqrt[6]{a}$; $a^{\frac{3}{7}} = \sqrt[7]{a^3}$; $a^{\frac{1}{4}} b^{\frac{3}{4}} = \sqrt[4]{ab^3}$; $2x^{\frac{1}{6}} y^{\frac{5}{6}} = 2\sqrt[6]{xy^5}$;

$5x^{\frac{1}{2}} y^{\frac{3}{4}} = 5x^{\frac{2}{4}} y^{\frac{3}{4}} = 5\sqrt[4]{x^2 y^3}$; $6x^{\frac{3}{5}} y^{\frac{1}{10}} z^{\frac{3}{5}} = 6x^{\frac{6}{10}} y^{\frac{1}{10}} z^{\frac{6}{10}} = 6\sqrt[10]{x^6 y z^6}$;

$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$; $81^{\frac{3}{4}} = \sqrt[4]{81^3} = 3^3 = 27$.

117. The signification of a negative index is also determined on the principle that the relation $x^m \times x^n = x^{m+n}$ is to be maintained when either (or both) of the indices is negative.

Let n become $-p$, then by the supposition

$$x^m \times x^{-p} = x^{m-p};$$

but by Art. 113 $$\frac{x^m}{x^p} = x^{m-p};$$

$$\therefore\ x^m \times x^{-p} = \frac{x^m}{x^p},$$

which proves that to multiply by x^{-p} is the same as to divide by x^p, or in other words that $x^{-p} = \frac{1}{x^p}$.

Exs. $x^{-2} = \frac{1}{x^2}$; $3a^{-3}b^{-4} = \frac{3}{a^3b^4}$; $4ab^{-1}c^{-2} = \frac{4a}{bc^2}$; $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$; $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$; $\frac{10a^{-1}}{b^{-2}c^{-4}} = \frac{10b^2c^4}{a}$; $\frac{5x^{-1}y^{-2}z}{3a^2b^{-2}c^{-1}} = \frac{5b^2cz}{3a^2xy^2}$.

118. When the index of any quantity is both fractional and negative, the negative index must first be removed by taking the reciprocal of the quantity, and afterwards the root taken as in Art. 116.

Exs. $16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}$; $64^{-\frac{5}{6}} = \frac{1}{64^{\frac{5}{6}}} = \frac{1}{\sqrt[6]{64^5}} = \frac{1}{32}$;

$125^{-\frac{2}{3}} = \frac{1}{125^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{125^2}} = \frac{1}{5^2} = \frac{1}{25}$; $a^{-\frac{2}{5}}b^{-\frac{2}{5}} = \frac{1}{\sqrt[5]{a^2b^2}}$;

$3x^{-1}y^{-\frac{1}{2}}z^{-\frac{3}{4}} = \frac{3}{x\sqrt[4]{y^2z^3}}$.

119. *To prove that* $a^0 = 1$.

Since the relation $x^{m-n} = \frac{x^m}{x^n}$ is true for all values of m

and n, let $n=m$; then

$$x^{m-m}=\frac{x^m}{x^m};$$

$$\therefore\ x^0=1.$$

120. In the solution of questions the student should observe the difference in the results of Arts. 112 and 114; thus e.g. $a^3\times a^4=a^7$, but $(a^3)^4=a^{12}$;

$x^{\frac{3}{2}}\times x^{\frac{1}{2}}=x$, but $(x^{\frac{3}{2}})^{\frac{1}{2}}=x^{\frac{3}{4}}$.

Exercise XXIV.

Express the following quantities with roots so as to remove all the fractional indices:

1. $a^{\frac{1}{2}}+a^{\frac{3}{4}}+4a^{\frac{5}{6}}$.
2. $a^{\frac{3}{4}}b^{\frac{1}{4}}+2a^{\frac{5}{6}}b^{\frac{1}{6}}+3ab^{\frac{3}{5}}$.
3. $x^{\frac{1}{2}}y^{\frac{1}{4}}+3x^{\frac{1}{3}}y^{\frac{5}{6}}-2x^{\frac{1}{5}}y^{\frac{3}{10}}$.

Remove all the roots from the following by the use of fractional indices:

4. $\sqrt{x}+\sqrt[3]{xy}+2\sqrt[4]{x^2y^3}$.
5. $\sqrt[6]{a^2b}+2\sqrt[5]{a^{10}b^3}-3\sqrt[4]{ab^3}$.
6. $\sqrt{4a^2b^3}-\sqrt[3]{27a^4b}+4\sqrt[10]{a^5b^7}$.

Express the following without denominators and root-signs by the use of negative and fractional indices:

7. $\frac{a^2}{b^3}+\frac{3ac^2}{b^3}-\frac{2b^3}{a^2c}$.
8. $\frac{3x}{y^3z}-\frac{4y^2}{x^3z^2}+\frac{x^3y}{z^4}$.
9. $\frac{3a}{\sqrt{x}}+\frac{5}{a^2\sqrt{x^3}}-\frac{6a}{b\sqrt{xy^3}}$.
10. $\frac{4\sqrt{x}}{\sqrt[3]{y}}-\frac{2\sqrt[3]{xy^2}}{\sqrt[4]{z^3}}-\frac{3\sqrt[4]{x}}{y\sqrt{z}}$.

Remove all denominators and negative signs from the indices of the following:

11. $a^{-2}bc^{-3}+2a^{-1}b^3c^{-2}-4a^{-3}b$.
12. $5a^{\frac{1}{2}}b^{-2}-7a^{\frac{5}{2}}b^{-1}+12a^{\frac{3}{4}}b^{-\frac{1}{2}}$.

13. $6x^{-\frac{1}{2}}y^{-1}z^{3}-8x^{-\frac{3}{4}}y^{\frac{1}{2}}+x^{-2}y^{\frac{1}{3}}z^{-\frac{1}{4}}$.

14. $\dfrac{3x^{-\frac{1}{2}}y^{\frac{1}{3}}}{z^{-\frac{1}{2}}}-\dfrac{4x^{-\frac{1}{2}}y^{-2}}{z^{-1}}+\dfrac{x^{-1}y^{\frac{3}{4}}}{2z^{-\frac{1}{4}}}$.

Simplify the following quantities:

15. $16^{\frac{1}{2}}$. 16. $27^{\frac{2}{3}}$. 17. $9^{\frac{3}{2}}$. 18. $81^{\frac{3}{4}}$.

19. $125^{\frac{2}{3}}$. 20. $(64a^{12})^{\frac{1}{6}}$. 21. $(243x^{10})^{\frac{2}{5}}$.

22. $(36x^{2}y^{4}z^{6})^{\frac{1}{2}}$. 23. $(32x^{5}y^{10}z^{20})^{\frac{3}{5}}$. 24. $25^{-\frac{1}{2}}$.

25. $1728^{-\frac{1}{3}}$. 26. $8^{-\frac{2}{3}}$. 27. $625^{-\frac{1}{4}}$.

28. $343^{-\frac{1}{3}}$. 29. $(216a^{6})^{-\frac{1}{3}}$. 30. $(100a^{2}b^{-4})^{-\frac{1}{2}}$.

31. $(16x^{-4}y^{8})^{-\frac{3}{4}}$. 32. $(x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{\frac{1}{6}})^{\frac{6}{5}}$.

33. $(x^{3}y^{\frac{1}{2}}z^{-\frac{2}{3}})^{15}$. 34. $x^{2}y^{\frac{1}{2}}\times(xy)^{15}$.

35 $(a^{\frac{1}{2}})^{-3}\times a^{\frac{3}{2}}$. 36. $\sqrt{(x^{\frac{1}{2}}y^{\frac{1}{2}})^{3}x^{-2}y^{-3}}$.

37. $\{(x^{\frac{1}{2}}y^{\frac{1}{3}})^{3}x^{-1}y^{-\frac{2}{3}}z^{\frac{1}{2}}\}^{4}$. 38. $3^{-\frac{1}{2}}\times\sqrt[3]{81\times5^{-\frac{3}{2}}}\times\sqrt{5}$.

39. $b^{\frac{1}{4}}\sqrt{\{(a^{\frac{1}{3}}b^{\frac{1}{2}}c)^{-12}a^{2}b^{-\frac{1}{2}}\}}$.

40. $\sqrt[5]{128}\times\left(\dfrac{1}{2}\right)^{\frac{2}{1\cdot6}}\times\sqrt[4]{\dfrac{1}{25}}\div360^{-\frac{1}{2}}$.

41. $\dfrac{1}{\sqrt[4]{a}}\times\{\sqrt[3]{a^{-\frac{3}{2}}b^{-4}}(a^{\frac{1}{2}}b^{-2})^{-\frac{1}{3}}\}^{-\frac{1}{2}}$.

42. If $3^{x}=\sqrt[3]{9}$, what is x?

43. Write down the quotient of
$$(x^{-3}a^{6}b^{9}+64b^{3})\div(x^{-1}a^{2}b^{3}+4b).$$

44. Resolve into factors
$$8x-10x^{\frac{3}{4}}y^{\frac{1}{4}}-3x^{\frac{1}{2}}y^{\frac{3}{4}}.$$

45. Extract the square root of
$$\frac{x}{y}\left(2+\frac{x}{y}\right)-\frac{y}{x}\left(2-\frac{y}{x}+\frac{x}{y}\right).$$

CHAPTER XXV.

SURDS.

121. WHEN the root of any quantity cannot be exactly determined, it is called a *Surd*, or is said to be *irrational*: thus

$$\sqrt{3},\ \sqrt[3]{4},\ \sqrt{a^3},\ \sqrt[3]{a^2},$$

are surds or irrational quantities.

122. Sometimes quantities are written in the form of surds, although capable of being expressed as rational quantities: thus

$$\sqrt{25},\ \sqrt[3]{27},\ \sqrt{9a^4},\ \sqrt[4]{16x^4},$$

are irrational only in form, because they are respectively equal to 5, 3, $3a^2$, $2x$.

123. Surds may often be changed in form by removing from under the root some factor whose root can be obtained: thus

$$\sqrt{18} = \sqrt{9\times 2} = 3\sqrt{2};$$

$$\sqrt[3]{375} = \sqrt[3]{125\times 3} = 5\sqrt[3]{3};$$

$$\sqrt{50a^3} = \sqrt{25a^2\times 2a} = 5a\sqrt{2a};$$

$$\sqrt[3]{81x^4y^6} = \sqrt[3]{27y^6\times 3x^3} = 3y^2\sqrt[3]{3x^3}.$$

124. When surds are connected by signs of addition or subtraction, they can be reduced to one quantity if they can all be expressed as multiples of the same surd.

Ex. 1. $\sqrt{75}+2\sqrt{147}-3\sqrt{48} = 5\sqrt{3}+14\sqrt{3}-12\sqrt{3}$

$= 7\sqrt{3}$. Ans.

125. **Surds having the same index are multiplied together by taking the product of the quantities under the signs, and retaining the common index; thus**

$$\sqrt{6}\times\sqrt{5}=\sqrt{30};\quad 2\sqrt{2}\times 3\sqrt{7}=6\sqrt{14};$$

$$2\sqrt{15}\times 7\sqrt{10}=14\sqrt{150}=70\sqrt{6}.$$

Ex. 2. Multiply $3\sqrt{6}+4\sqrt{2}$ by $2\sqrt{6}-3\sqrt{2}$.

$$\begin{array}{r} 3\sqrt{6}+4\sqrt{2} \\ 2\sqrt{6}-3\sqrt{2} \\ \hline 36+8\sqrt{12} \\ \quad -9\sqrt{12}-24 \\ \hline 12-\sqrt{12}. \end{array}$$

Hence the product is $12-\sqrt{12}$, or $12-2\sqrt{3}$. Ans.

126. The denominator of a fraction is said to be *rationalised* when the form of the fraction is so changed that all surds are removed from the denominator.

When there are only two terms in the denominator it may always be rationalised without altering the value of the fraction by multiplying both numerator and denominator by a quantity containing terms the same as the denominator but connected by the opposite sign.

Ex. 3. Rationalise the denominators of

$$\frac{6}{2-\sqrt{3}} \text{ and } \frac{3+\sqrt{5}}{7+2\sqrt{5}}.$$

In the first case we multiply numerator and denominator by $2+\sqrt{3}$; thus

$$\frac{6}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{6(2+\sqrt{3})}{4-3}=6(2+\sqrt{3}).$$

In the other fraction the multiplier is $7-2\sqrt{5}$; thus

$$\frac{3+\sqrt{5}}{7+2\sqrt{5}}=\frac{3+\sqrt{5}}{7+2\sqrt{5}}\times\frac{7-2\sqrt{5}}{7-2\sqrt{5}}=\frac{21+\sqrt{5}-10}{49-20}=\frac{1}{29}(11+\sqrt{5}).$$

Ex. 4. Divide $4\sqrt{2}+\sqrt{6}$ by $2\sqrt{2}-\sqrt{6}$.

$$(4\sqrt{2}+\sqrt{6})\div(2\sqrt{2}-\sqrt{6})$$

$$=\frac{4\sqrt{2}+\sqrt{6}}{2\sqrt{2}-\sqrt{6}}=\frac{16+6\sqrt{12}+6}{8-6}=\frac{22+12\sqrt{3}}{2}=11+6\sqrt{3}.$$ **Ans.**

127. *The square root of a quantity cannot be partly rational and partly irrational.*

If possible, let

$$\sqrt{n}=a+\sqrt{m}.$$

Then, by squaring

$$n=a^2+2a\sqrt{m}+m;$$

$$\therefore\ 2a\sqrt{m}=n-a^2-m,$$

and

$$\sqrt{m}=\frac{n-a^2-m}{2a};$$

i.e. $\sqrt{m}$ is equal to a rational quantity, which is contrary to the supposition.

128. *If any two quantities, partly rational and partly irrational, be equal to one another, the rational parts of the two are equal, and also the irrational.*

Let $x+\sqrt{y}=a+\sqrt{b}$; then shall $x=a$, and $\sqrt{y}=\sqrt{b}$; for if x be not equal to a, let $x=a+m$,

then

$$a+m+\sqrt{y}=a+\sqrt{b};$$

$$\therefore\ m+\sqrt{y}=\sqrt{b},$$

that is, $\sqrt{b}$ is partly rational and partly irrational, which is

impossible; therefore $x=a$,

and $$\sqrt{y}=\sqrt{b}.$$

129. *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$; *then also* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$,

Since $$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y},$$

by squaring

$$a+\sqrt{b}=x+y+2\sqrt{xy};$$

therefore by Art. 128

$$a=x+y, \text{ and } \sqrt{b}=2\sqrt{xy};$$

$$\therefore a-\sqrt{b}=x+y-2\sqrt{xy},$$

and $$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

130. *To find the square root of any binomial surd* $a+\sqrt{b}$.

Let $$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y} \quad \ldots\ldots(1),$$

then by Art. 129

$$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y};$$

$\therefore$ by multiplication

$$\sqrt{a^2-b}=x-y \quad \ldots\ldots(2).$$

Again, by squaring (1),

$$a+\sqrt{b}=x+y+2\sqrt{xy};$$

$\therefore$ by Art. 128 $$a=x+y \quad \ldots\ldots(3).$$

By combining (2) and (3) we can find x and y, and by substituting in (1), we obtain the root of $a+\sqrt{b}$.

Ex. 5. Find the square root of $57+28\sqrt{2}$.

Let $\sqrt{57+28\sqrt{2}}=\sqrt{x}+\sqrt{y}$ ..... (1);

then $\sqrt{57-28\sqrt{2}}=\sqrt{x}-\sqrt{y}$;

$\therefore\ \sqrt{3249-1568}=x-y,$

$\sqrt{1681}=x-y,$

$41=x-y$ ..... (2);

but from (1) $57=x+y$ ..... (3).

By adding (2) and (3),

$2x=98,\quad x=49,\quad$ and $\quad y=8.$

Substitute in (1); then

$$\sqrt{57+28\sqrt{2}}=\sqrt{49}+\sqrt{8}=7+2\sqrt{2}.\quad \text{Ans.}$$

131. Ex. 6. Simplify $\dfrac{9-\sqrt{6}}{\sqrt{22-8\sqrt{6}}}$.

Let $\sqrt{22-8\sqrt{6}}=\sqrt{x}-\sqrt{y},$

then $\sqrt{22+8\sqrt{6}}=\sqrt{x}+\sqrt{y}$;

$\therefore\ \sqrt{484-384}=x-y$;

$\therefore\ x-y=10.$

Also $x+y=22$;

$\therefore\ x=16;\ y=6,$

and $\sqrt{22-8\sqrt{6}}=\sqrt{16}-\sqrt{6}=4-\sqrt{6}.$

Hence $$\frac{9-\sqrt{6}}{\sqrt{22-8\sqrt{6}}}=\frac{9-\sqrt{6}}{4-\sqrt{6}}=\frac{9-\sqrt{6}}{4-\sqrt{6}}\times\frac{4+\sqrt{6}}{4+\sqrt{6}}$$

$$=\frac{30+5\sqrt{6}}{10}=\frac{1}{2}(6+\sqrt{6}).\quad \text{Ans.}$$

Exercise XXV.

Simplify the following quantities:

1. $\sqrt{72}$.
2. $\sqrt{80}$.
3. $\sqrt{343}$.
4. $\sqrt{1000}$.
5. $\sqrt{75a^3}$.
6. $\sqrt{384x^5y^2}$.
7. $\sqrt[3]{16}$.
8. $\sqrt[3]{135}$.
9. $\sqrt[3]{3456x^6y^4}$.
10. $\sqrt[4]{162x^5y^{13}}$.
11. $4\sqrt{50}$.
12. $7\sqrt{32a^3b^2}$.
13. $20a\sqrt{27a^3b^4}$.
14. $5\sqrt{605a^3x^5}$.
15. $10\sqrt[3]{375}$.
16. $3\sqrt[4]{1250}$.
17. $3\sqrt{2}+\sqrt{8}+3\sqrt{50}$.
18. $8\sqrt{3}+2\sqrt{12}-3\sqrt{48}$.
19. $2\sqrt{75}+\sqrt{48}-\sqrt{147}$.
20. $4\sqrt{80}-2\sqrt{125}+3\sqrt{320}$.
21. $2\sqrt[3]{4}-5\sqrt[3]{32}+3\sqrt[3]{108}$.
22. $\sqrt[3]{40}+2\sqrt[3]{625}-3\sqrt[3]{320}$.

Multiply

23. $3\sqrt{5}$ by $2\sqrt{3}$.
24. $7\sqrt{15}$ by $2\sqrt{5}$.
25. $5\sqrt{14}$ by $6\sqrt{21}$.
26. $7\sqrt{2}-3\sqrt{3}$ by $4\sqrt{2}+\sqrt{3}$.
27. $4\sqrt{5}+3\sqrt{3}$ by $\sqrt{5}+2\sqrt{3}$.
28. $6\sqrt{10}-2\sqrt{3}$ by $2\sqrt{10}-\sqrt{3}$.
29. $4\sqrt{6}-2\sqrt{2}$ by $3\sqrt{2}+\sqrt{6}$.
30. $11\sqrt{3}+2\sqrt{10}$ by $\sqrt{6}-\sqrt{5}$.
31. $3\sqrt{5}+7\sqrt{2}-2\sqrt{3}$ by $4\sqrt{5}+\sqrt{3}$.
32. $4\sqrt{6}-\sqrt{2}+2\sqrt{3}$ by $2\sqrt{2}-\sqrt{3}$.
33. $\frac{\sqrt{5}}{4}-\frac{2\sqrt{2}}{5}+\sqrt{10}$ by $4\sqrt{5}-\sqrt{10}$.

34. $\frac{\sqrt{3}}{2}-\frac{\sqrt{10}}{3}+\frac{2}{\sqrt{3}}$ by $\sqrt{6}-\frac{\sqrt{5}}{4}$.

35. $\sqrt[3]{3}+\sqrt[3]{4}$ by $\sqrt[3]{9}+\sqrt[3]{2}$.

36. $2\sqrt[3]{2}-\sqrt[3]{12}$ by $3\sqrt[3]{4}-\sqrt[3]{3}$.

Rationalise the denominator of

37. $\frac{4}{\sqrt{3}}$.

38. $\frac{3\sqrt{2}}{\sqrt{5}}$.

39. $\frac{4}{3-\sqrt{5}}$.

40. $\frac{6}{4+\sqrt{7}}$.

41. $\frac{4+\sqrt{5}}{3-\sqrt{5}}$.

42. $\frac{7-2\sqrt{2}}{4+3\sqrt{2}}$.

43. $\frac{3+2\sqrt{5}}{6-2\sqrt{5}}$.

44. $\frac{3+\sqrt{5}}{4-\sqrt{10}}$.

45. $\frac{2\sqrt{3}+4\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

46. $\frac{7\sqrt{5}-3}{\sqrt{10}-\sqrt{2}}$.

47. $\frac{\sqrt{20}-\sqrt{12}+\sqrt{10}}{\sqrt{5}-\sqrt{3}}$.

48. $\frac{2\sqrt{3}+\sqrt{8}-\sqrt{10}}{2\sqrt{10}-\sqrt{3}}$.

Simplify

49. $\frac{(7-2\sqrt{5})(5+\sqrt{7})(31+13\sqrt{5})}{(6-2\sqrt{7})(3+\sqrt{5})(11+4\sqrt{7})}$.

50. $\frac{5+\sqrt{7}}{4+\sqrt{7}}+\frac{5-\sqrt{7}}{4-\sqrt{7}}$.

Divide

51. 7 by $\sqrt{10}+\sqrt{3}$.

52. $3+\sqrt{5}$ by $\sqrt{5}-2$.

53. $4+\sqrt{2}$ by $3-2\sqrt{2}$.

54. $8\sqrt{3}-5\sqrt{2}$ by $7\sqrt{3}+\sqrt{2}$.

55. $3\sqrt{5}+2\sqrt{3}$ by $4\sqrt{3}-3\sqrt{5}$.

56. $\sqrt{60}-\sqrt{45}+\sqrt{8}$ by $\sqrt{15}+\sqrt{2}$.

Find the square root of

57. $19+8\sqrt{3}$.

58. $41-12\sqrt{5}$.

59. $14+4\sqrt{10}$.

60. $22-4\sqrt{10}$.

61. $49+20\sqrt{6}$.

62. $44-24\sqrt{2}$.

63. $10-2\sqrt{21}$.

64. $31-4\sqrt{55}$.

65. $110-60\sqrt{2}$.

66. $66-24\sqrt{6}$.

Simplify

67. $\dfrac{1}{\sqrt{3+2\sqrt{2}}}$.

68. $\dfrac{2}{\sqrt{6-4\sqrt{2}}}$.

69. $\dfrac{1}{\sqrt{51+14\sqrt{2}}}$.

70. $\dfrac{2+\sqrt{3}}{\sqrt{7-4\sqrt{3}}}$.

71. $\dfrac{\sqrt{2}-1}{\sqrt{15-10\sqrt{2}}}$.

72. $\dfrac{\sqrt{6}+1}{\sqrt{10-4\sqrt{6}}}$.

73. **Prove that**

$$\frac{\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}}{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}} \times \frac{3^{\frac{1}{3}}+2^{\frac{1}{3}}}{3^{\frac{1}{3}}-2^{\frac{1}{3}}}=5.$$

74. Divide $7+\sqrt{3}$ by $\sqrt{3}-1$; add 2 to the quotient, and extract the square root of the sum.

CHAPTER XXVI.

ARITHMETICAL PROGRESSION.

132. QUANTITIES are said to be in *Arithmetical Progression* when they increase or decrease by a *common difference.*

Thus each of the following series forms an Arithmetical Progression:

$2,\ 5,\ 8,\ 11\ldots\ldots$

$5,\ 1,\ -3,\ -7\ldots\ldots$

$a,\ a+d,\ a+2d,\ a+3d\ldots\ldots$

$a,\ a-d,\ a-2d,\ a-3d\ldots\ldots$

133. *To find the* n^{th} *term of a series in Arithmetical Progression,*

$$a,\ a+d,\ a+2d\ldots\ldots$$

Since each term is formed from the preceding term by adding to it the common difference, the series is

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ldots\ldots$$

in which we observe that the *coefficient of d is always less by one than the number of the term;* hence if l be the n^{th} term

$$l=a+(n-1)d.$$

134. The student may by the above formula determine any term in a given series; thus suppose we want the 25th term of the series 2, 5, 8..., we have

$$a=2,\ d=3,\ n=25\ ;$$

$$\therefore\ l=2+(25-1)\,3=2+72=74.$$

But it will help to a better understanding of the rule if any required term be found independently of the formula: thus the series

$$2,\ 5,\ 8,\ 11,...$$

may be written

$$2,\ 2+3,\ 2+2\cdot3,\ 2+3\cdot3,.....$$

in which we observe as before that the coefficient of the common difference 3 is always less by one than the number of the term: hence the 25th term is

$$2+24\cdot3=74.$$

135. The beginner sometimes has a difficulty in determining the common difference when the first term is positive, and the second negative: he should observe that the difference is always the second term *minus* the first term: thus in the series

$$3,\ -2,\ -7.....;\qquad d=-2-3=-5.$$

$$\frac{1}{5},\ -\frac{1}{3},\ -\frac{13}{15}.....;\quad d=-\frac{1}{3}-\frac{1}{5}=-\frac{8}{15}.$$

$$-2,\ -\frac{6}{5},\ -\frac{2}{5}.....;\qquad d=-\frac{6}{5}+2=\frac{4}{5}.$$

136. *To find the sum of* n *terms of an Arithmetical Progression,*

$$a,\ a+d,\ a+2d,.....$$

Let l be the n^{th} term as in Art. 133, and let s be the sum of the terms, then

$$s=a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l;$$

$\therefore$ by reversing the series we have also

$$s=l+(l-d)+(l-2d)+\ldots\ldots+(a+2d)+(a+d)+a.$$

Adding together these two lines we obtain

$$2s=(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l)+(a+l)+(a+l)$$
$$=n(a+l), \text{ since there are } n \text{ terms;}$$

$$\therefore\ s=\frac{n}{2}(a+l).$$

Another form may be given to this value of s, by substituting for l the value obtained for it in Art. 133,

$$s=\frac{n}{2}(a+a+\overline{n-1}d),$$

$$\therefore\ s=\frac{n}{2}(2a+\overline{n-1}d).$$

137. For the sake of reference we place together the three formulæ of this chapter:

$$l=a+(n-1)\,d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$s=(a+l)\frac{n}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$s=\{2a+(n-1)\,d\}\frac{n}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The student will observe that of the five letters employed four are found in each of the above formulæ; hence when any three of the letters are given, we can, by selecting from these formulæ, determine the other two.

138. **Ex. 1. Find the last term and the sum of the series**

$$1,\ 2\tfrac{3}{4},\ 4\tfrac{1}{2} \ldots\ldots \text{ to 21 terms.}$$

The series may be written

$$1,\ 1+1\tfrac{3}{4},\ 1+2\,.\,1\tfrac{3}{4}, \ldots\ldots$$

$$\therefore \text{ the last term} = 1+20\times 1\tfrac{3}{4} = 36.$$

Hence $s = 1+2\tfrac{3}{4}+4\tfrac{1}{2}+\ldots\ldots+32\tfrac{1}{2}+34\tfrac{1}{4}+36.$

Also $s = 36+34\tfrac{1}{4}+32\tfrac{1}{2}+\ldots\ldots+4\tfrac{1}{2}+2\tfrac{3}{4}+1;$

$$\therefore\ 2s = 37+37+\ldots\ldots+37+37 \text{ to 21 terms}$$

$$= 37\times 21 = 777;$$

$$\therefore\ s = 388\tfrac{1}{2}.$$

Thus the last term is 36, and the sum $388\tfrac{1}{2}$.

Ex. 2. Insert 4 arithmetical means between -5 and 15.

Since there are to be four means, there will altogether be six terms: hence we have $a=-5$, $l=15$, $n=6$; to find d.

By (1) of Art. 137 $l = a+(n-1)\,d,$

or $15 = -5+5d;$

$$\therefore\ d = 4.$$

Hence the required terms are -1, 3, 7, 11. Ans.

Ex. 3. In an A. P. the first term is 3, and the last term 27; also the sum of the terms is 495. How many terms are there in the series?

We have $a=3$, $l=27$, $s=495$; to find n.

These are the letters in formula (2) of Art. 157; hence by substitution,

$$495 = (3+27)\,\frac{n}{2}$$

$$= 15n;$$

$$\therefore\ n = 33. \quad \text{Ans.}$$

Ex. 4. Find how many terms of the series 2, 5, 8, ... will amount to 610.

We have $a=2$, $d=3$, $s=610$; to find n.

Substitute in (3) of Art. 137,

then $$610=\{4+3(n-1)\}\frac{n}{2}$$

$$=(1+3n)\frac{n}{2};$$

$$\therefore\ 3n^2+n-1220=0.$$

From which $n=20$. Ans.

Ex. 5. The sum of five terms of an A. P. is 15, and the sum of their squares is 85: find them.

Let $a-2b$, $a-b$, a, $a+b$, $a+2b$, be the terms, then

$$a-2b+a-b+a+a+b+a+2b=15 \ldots\ldots (1),$$

and $$(a-2b)^2+(a-b)^2+a^2+(a+b)^2+(a+2b)^2=85 \ldots\ldots (2).$$

From (1) $5a=15$, or $a=3$.

From (2) $5a^2+10b^2=85$,

$10b^2=40$, or $b=\pm 2$.

Hence the series is -1, 1, 3, 5, 7. Ans.

Exercise XXVI.

1. Find the 10th and 21st terms in the series 1, 4, 7....

2. Find the 16th and 22nd terms in the series 2, 6, 10....

3. Find the 11th and 25th terms in the series 4, $5\frac{1}{2}$, 7....

4. Find the 15th and 23rd terms in the series 3, 1, −1....

5. Find the 9th and 101st terms in the series 1, $-\frac{1}{2}$, −2....

Determine the last term and the sum of the following series :

6. 1, 5, 9 ... to 21 terms.
7. 3, 6, 9 ... to 14 terms.
8. 10, 13, 16 ... to 100 terms.
9. 4, $5\frac{1}{4}$, $6\frac{1}{2}$ ... to 13 terms.
10. 2, $3\frac{1}{2}$, 5 ... to 51 terms.
11. 1, $1\frac{3}{8}$, $1\frac{3}{4}$ ... to 97 terms.
12. 5, $5\frac{3}{4}$, $6\frac{1}{2}$ ... to 49 terms.
13. 10, 3, −4 ... to 17 terms.
14. 20, $16\frac{1}{4}$, $12\frac{1}{2}$ ... to 101 terms.
15. $2\frac{1}{2}$, 0, $-2\frac{1}{2}$ ... to 38 terms.
16. 1, $-\frac{1}{5}$, $-1\frac{2}{5}$ ... to 11 terms.
17. Insert 5 arithmetical means between 17 and 26.
18. Insert 4 arithmetical means between 7 and −3.
19. Insert 5 arithmetical means between $4\frac{1}{2}$ and −3.
20. Insert 3 arithmetical means between $7\frac{3}{4}$ and $-23\frac{1}{4}$.
21. Insert 2 arithmetical means between −2 and $3\frac{1}{4}$.
22. Insert 3 arithmetical means between $\frac{11}{15}$ and $-\frac{3}{5}$.
23. The last term of the series 4, 3, 2 ... is −74; find the number of terms.

24. How many terms of the series 3, 4, 5, ... must be taken to make 52 ?

25. The sum of 5 terms of an Arithmetical series is 55, and the fifth term is 17 ; find the series.

26. The sum of 5 terms of a series beginning with 10 is -20 ; find the last term.

27. If the first term of a series be 7, difference 2, and sum 1927 ; find the number of terms.

28. How many terms of the series 5, $2\frac{1}{2}$, 0 ... must be taken to make -130 ?

29. The third term of a series in A. P. is 6, and the sixth term is 18 ; find the series.

30. The fourth term of an A. P. is 15 ; the eighth term 31 ; how many terms must be taken to make the sum amount to 1378 ?

31. The sum of five numbers in A. P. is 25, and the sum of their squares is $147\frac{1}{2}$; find them.

32. The sixth term of an Arithmetical series is -8, and the sum of six terms is -3 ; find the series.

33. In a series whose common difference is $1\frac{1}{4}$, the sum of the first and last terms is $5\frac{3}{4}$, and the sum of the means is $17\frac{1}{4}$; find the number of terms and the first term.

34. The sum of 4 terms of an A. P. is $10\frac{1}{2}$, and the sum of their squares is $35\frac{3}{8}$; find the terms.

35. Find the sum of the first ten numbers which are divisible by 4.

36. The fourth term of an A. P. is a, and the seventh term b ; find the sum of 10 terms.

37. The sum of the first five terms of an A. P. is 120, and the sum of the next four terms is 98: find the series.

38. Sum the series $1-3+5-7+\ldots$ to $2n$ terms.

39. A and B have to walk a distance of 27 miles; A starts at $2\frac{1}{2}$ miles an hour and increases his pace by a quarter of a mile every hour; B starts at 5 miles an hour but falls off at the rate of half a mile every hour; find who will finish the distance first and by what length of time.

40. If $S_1, S_2, S_3 \ldots S_m$ be the sums of n terms of $2n$ Arithmetical series which have the same first term, and common differences d, $2d$, $3d$ ... $2nd$, shew that

$$(S_2+S_4+\ldots+S_{2n})-(S_1+S_3+\ldots+S_{2n-1})=\frac{n^2(n-1)\,d}{2}.$$

CHAPTER XXVII.

GEOMETRICAL PROGRESSION.

139. QUANTITIES are said to be in *Geometrical Progression* when any term may be derived from that preceding it by multiplying it by a *common ratio*.

Thus each of the following series forms a Geometrical Progression:

$$4,\ 12,\ 36,\ 108\ldots$$

$$6,\ -3,\ \frac{3}{2},\ -\frac{3}{4}\ldots$$

$$a,\ ar,\ ar^2,\ ar^3\ldots.$$

140. From the above definition we see that when the series is given the ratio is found by dividing the second term by the first, or any term by that which precedes it.

Thus in 4, 12, 36 ..., the ratio is $\frac{12}{4}=3$,

..... 6, -3, $\frac{3}{2}$..., ...$-\frac{3}{6}=-\frac{1}{2}$,

..... a, ar, ar^2..., ... $\frac{ar}{a}=r$.

141. *To find the last term of a Geometrical Progression whose first term is* a, *and common ratio* r.

It is evident that the series is a, ar, ar^2, ar^3..., in which we observe that the index of r in any term is less by one than the number of the term: hence if l be the n^{th} term, $l=ar^{n-1}$.

Ex. 1. Find the 8th term in the series 7, 14, 28, ...

The series may be written 7, 7×2, 7×2^2, ...

Hence the 8th term $=7\times2^7=7\times128=896$. Ans.

Ex. 2. Find the 6th term of 3, 1, $\frac{1}{3}$, ...

The series may be written 3, $3\times\frac{1}{3}$, $3\times\frac{1}{3^2}$, ...

Hence the 6th term is $3\times\frac{1}{3^5}=\frac{1}{81}$. Ans.

142. *To find the sum of* n *terms of a Geometrical Progression whose first term is* a, *and common ratio* r.

Let s represent the sum of n terms, then

$$s=a+ar+ar^2+\ldots+ar^{n-1}.$$

Multiplying both sides by r we have

$$rs=ar+ar^2+\ldots+ar^{n-1}+ar^n;$$

then by subtraction $rs-s=ar^n-a$,

$$s(r-1)=a(r^n-1),$$

$$s=a\cdot\frac{r^n-1}{r-1}.$$

This formula may also be written in another form; by substituting the value $l=ar^{n-1}$, we obtain

$$s=\frac{lr-a}{r-1}.$$

Ex. 3. Find the sum of 7 terms of the series 5, 10, 20.....

$$s=5+5\times2+5\times2^2+\ldots+5\times2^6.$$

Multiply by 2,

$$2s=\quad 5\times2+5\times2^2+\ldots+5\times2^6+5\times2^7;$$

$\therefore$ by subtraction, $s=5\times2^7-5$

$$=640-5=635. \text{ Ans.}$$

Ex. 4. Find the sum of 8 terms of the series

$$2-1+\frac{1}{2}-\ldots.$$

$$s=2-2\times\frac{1}{2}+2\times\frac{1}{2^2}-2\times\frac{1}{2^3}\ldots\ldots-2\times\frac{1}{2^7};$$

$$\frac{1}{2}s=\quad 2\times\frac{1}{2}-2\times\frac{1}{2^2}+2\times\frac{1}{2^3}\ldots\ldots+2\times\frac{1}{2^7}-2\times\frac{1}{2^8};$$

$\therefore$ by addition, $\frac{3}{2}s=2-2\times\frac{1}{2^8}$

$$=2-\frac{1}{128}=1\tfrac{127}{128}.$$

$$\therefore s=1\tfrac{21}{64}. \text{ Ans.}$$

143. When r is a proper fraction, the value of r^n decreases as n increases: when n is increased without limit (that is, when the number of terms is carried *ad infinitum*)

the value of r^n becomes inappreciable and may be neglected: hence the expression $s=a\frac{r^n-1}{r-1}$ (Art. 142) becomes in this case

$$s=a\times\frac{-1}{r-1}=\frac{a}{1-r}.$$

This value is called the *limit* of the series, because while the sum never actually reaches this value, we may by taking more terms of the series make it approach to it as nearly as we please.

Ex. 5. Find the limit of the series $8+2+\frac{1}{2}+\ldots$

$$a=8,\quad r=\frac{2}{8}=\frac{1}{4}.$$

Hence the limit $=\frac{8}{1-\frac{1}{4}}=\frac{32}{3}=10\frac{2}{3}$. Ans.

144. We place together here the values which have been obtained for the last term and sum of a G. P.

$$l=ar^{n-1} \quad \ldots\ldots\ldots\ldots (1),$$

$$s=a\cdot\frac{r^n-1}{r-1} \quad \ldots\ldots\ldots\ldots (2),$$

$$s=\frac{lr-a}{r-1} \quad \ldots\ldots\ldots\ldots (3).$$

Also when the ratio is a proper fraction and the series taken *ad infinitum*,

$$s=\frac{a}{1-r} \quad \ldots\ldots\ldots\ldots (4).$$

145. Ex. 6. Insert 3 geometrical means between 7 and $34\frac{3}{16}$.

Since there are to be 3 means, the number of terms is 5; thus

$$a=7, \quad l=34\tfrac{3}{16}, \quad n=5.$$

Now by (1) of Art. 144, $l=ar^{n-1}$;

$$\therefore \ 34\tfrac{3}{16}=7\times r^4,$$

$$r^4=\frac{547}{16}\div 7=\frac{81}{16},$$

$$r=\frac{3}{2};$$

$$\therefore \text{ the means are } \frac{21}{2}, \ \frac{63}{4}, \ \frac{189}{8}. \quad \text{Ans.}$$

146. We shall now indicate by an example how the rule for reducing a recurring decimal to a vulgar fraction is obtained.

Ex. 7. Reduce ·2671671671 ... to a vulgar fraction.

Let F be the required fraction.

Now multiply F by such a power of 10 as will move the decimal point to the *end* of the first recurring period, and also by such a power of 10 as will move it to the *beginning* of this period, thus:

$$10000F=2671\cdot671671\ldots$$

$$10F=\quad 2\cdot671671\ldots$$

by subtraction, $\quad 9990F=2669$

$$F=\frac{2669}{9990}.$$

Exercise XXVII.

1. Find the 8th term of the series 3, 6, 12.....
2. Find the 6th term of the series 2, 6, 18.....
3. Find the 5th term of the series 10, -30, 90.....
4. Find the 8th term of the series 5, -10, 20.....
5. Find the 6th term of the series $4\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$.....
6. Find the 7th term of the series $\frac{2}{5}$, $-\frac{1}{5}$, $\frac{1}{10}$.....

Find the last term, and the sum, of the following series:

7. 8, 16, 32..... to 8 terms.
8. 4, -12, 36..... to 7 terms.
9. 1, $\frac{1}{3}$, $\frac{1}{9}$..... to 6 terms.
10. 2, $\frac{1}{2}$, $\frac{1}{8}$..... to 5 terms.
11. 3, -1, $\frac{1}{3}$..... to 7 terms.
12. $\frac{2}{5}$, $\frac{1}{2}$, $\frac{5}{8}$..... to 4 terms.

Find the sum of

13 10, 20, 40..... to 8 terms.
14. 6, 3, $1\frac{1}{2}$..... to 7 terms.
15. 4, -12, 36..... to 7 terms.
16. 1, $\frac{1}{2}$, $\frac{1}{4}$..... to infinity.

17. $\frac{2}{3}, \frac{1}{6}, \frac{1}{24}$ ..... to infinity.

18. $3, -\frac{2}{3}, \frac{4}{27}$ .....

19. $\frac{3}{4}, -\frac{1}{2}, \frac{1}{3}$ .....

20. $\frac{8}{15}, -\frac{1}{3}, \frac{5}{24}$ .....

21. Insert 2 geometrical means between 4 and $7\frac{1}{9}$.

22. Insert 4 geometrical means between 4 and $-\frac{1}{8}$.

23. Insert 3 geometrical means between $4\frac{1}{4}\frac{1}{4}$ and 2; also 5 between 36 and $771\frac{1}{4}\frac{1}{4}$.

24. Shew how to reduce the following recurring decimals to vulgar fractions: ·35757...; ·02138138...; $2{\cdot}05\dot{7}3\dot{0}$; $32{\cdot}0\dot{2}03\dot{5}$; $5{\cdot}6721721\ldots$.

25. The last term of a G.P. whose first term is 4, and ratio 2, is 512; find the number of terms.

26. The sum of an infinite series whose first term is 5, amounts to $7\frac{1}{2}$; find the common ratio.

27. The sum of a series whose first term is 2, and common ratio 3, amounts to 730; find the number of terms.

28. The third term of a geometrical series is $2\frac{1}{4}$, and the fifth term is $5\frac{1}{16}$; find the series.

29. Insert 4 geometrical means between 3 and $\frac{1}{81}$.

30. The sum of an infinite geometrical series is 12, and the second term is 27 times the fifth term; find the series.

31. The difference of two numbers is 48, and the arithmetic mean exceeds the geometric by 18; find the numbers.

32. If a, b, c are in G.P., shew that

$$\frac{3a+7b}{5a-7b}=\frac{3b+7c}{5b-7c}.$$

33. If a, b, c, d are in G.P., shew that their sum

$$=\sqrt{(a+b)^2+(c+d)^2+2(b+c)^2}.$$

34. The sum of three quantities in G.P. is $\frac{13}{6}$, and the sum of their reciprocals is $\frac{26}{3}$; find the quantities.

35. If S_n, S_n' be respectively the sum to n terms of the series

$$1+r+r^2+\ldots\ldots \quad \text{and} \quad 1+\frac{1}{r}+\frac{1}{r^2}+\ldots\ldots$$

shew that

$$\frac{S_1}{S_1'}+\frac{S_2}{S_2'}+\frac{S_3}{S_3'}+\ldots\ldots+\frac{S_n}{S_n'}=S_n.$$

36. If a hare starts with a certain velocity, which decreases suddenly every quarter of an hour in a geometrical progression whose ratio is $\frac{1}{10}$, and is rejoined in 44000 hours by a tortoise which starts with the hare at the constant rate of a yard in an hour; find the original velocity of the hare.

CHAPTER XXVIII.

HARMONICAL PROGRESSION.

147. Quantities are said to be in Harmonical Progression when their reciprocals are in Arithmetical Progression.

Thus

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ are in H. P., because 2, 4, 6 are in A. P.

$\frac{1}{a-b}, \frac{1}{a}, \frac{1}{a+b}$ .....$a-b, a, a+b$ .....

148. A general expression for the sum of an Harmonical Progression cannot be determined, but we can solve many questions with regard to quantities in H. P. by taking their reciprocals, and treating them as in A. P.

Ex. 1. Continue to two terms each way the Harmonical Progression

$$1, \frac{4}{5}, \frac{2}{3}.$$

The A. P. formed by the reciprocals is $1, \frac{5}{4}, \frac{3}{2}$, in which the common difference is $\frac{1}{4}$;

therefore the additional terms in the A. P. will be $\frac{1}{2}, \frac{3}{4}$, at the beginning, and $\frac{7}{4}$, 2 at the end of the series;

hence the required terms in the H. P. are 2, $1\frac{1}{2}$; and $\frac{4}{7}, \frac{1}{2}$. Ans.

Ex. 2. Insert 3 harmonical means between $\frac{1}{3}$ and $\frac{1}{6}$.

We first find 3 arithmetical means between 3 and 6; in the formula $l=a+\overline{n-1}.b$ (Art. 133)

$$l=6, \quad a=3, \quad n=5;$$

$$\therefore \ 6=3+4b, \text{ and } b=\frac{3}{4},$$

Hence the Arithmetical means are $3\frac{3}{4}$, $4\frac{1}{2}$, $5\frac{1}{4}$, and the Harmonical means are $\frac{4}{15}$, $\frac{2}{9}$, $\frac{4}{21}$. Ans.

149. *If* a, b, c *are in Harmonical Progression, to shew that* a : c :: a − b : b − c.

Since a, b, c are in H.P.,

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.;}$$

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b},$$

$$\frac{a-b}{a} = \frac{b-c}{c},$$

hence $$a : c :: a-b : b-c.$$

This proportion is sometimes given as the Definition of Harmonical Progression; it can then be deduced that the reciprocals of the quantities are in A. P.

150. A single Arithmetical mean between any two quantities a and b is called their Arithmetical mean, and is generally denoted by A: similarly the Geometrical mean and Harmonical mean are denoted by G and H. Their values may be found as follows:

(1) $a - A = A - b$ (Art. 132), $\therefore A = \frac{1}{2}(a+b)$.

(2) $a : G :: G : b$ (Art. 139), $\therefore G = \sqrt{ab}$.

(3) $a : b :: a - H : H - b$ (Art. 149), $\therefore H = \frac{2ab}{a+b}$.

EXERCISE XXVIII.

1. Continue to 3 terms the H. P. $1\frac{2}{3}$, 1, $\frac{5}{7}$.

2. Continue to 3 terms the H. P. 2, $1\frac{1}{3}$, $\frac{10}{13}$.

3. Continue to 3 terms each way $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$.

4. Continue to 2 terms each way $\frac{2}{9}$, $\frac{1}{6}$, $\frac{2}{15}$.

5. Insert 3 harmonical means between $4\frac{1}{2}$ and $1\frac{1}{2}$.

6. Insert 4 harmonical means between $\frac{12}{19}$ and 3; also between 3 and $\frac{3}{4}$.

7. The arithmetical mean between two numbers is $6\frac{1}{2}$, and the geometrical mean is 6; find the numbers.

8. The arithmetical and geometrical means between two numbers are 58 and 42 respectively; find the numbers.

9. Find two numbers whose geometrical and harmonical means are 5 and 4 respectively.

10. The geometric mean between two numbers is 6, and the harmonic mean is $5\frac{7}{13}$; find the numbers.

11. Find the 5th term of an H. P. whose first and third terms are 6 and 2.

12. Find the arithmetical, geometrical, harmonical means between 1 and 2, and continue the series which these means form to two more terms.

13. If a, g, h be the arithmetical, geometrical, and harmonical means between two given quantities, prove that g is the geometrical mean between a and h.

14. If a, b, c be three quantities such that a is the arithmetical mean between b and c, and c the harmonical mean between a and b, prove that b is the geometrical mean between a and c.

CHAPTER XXIX.

Scales of Notation.

151. In ordinary numbers each figure increases in value *tenfold* as we proceed from right to left; thus

$$8769 = 9 + 6 \times 10 + 7 \times 10^2 + 8 \times 10^3.$$

It is however only by general convention that 10 is chosen as the factor by which each figure increases, and if we make it clear what common factor we are using, it is quite open to us to write a number with any other figure as the understood multiplier instead of 10; thus if we choose 8, the number 7512 would represent

$$2 + 1 \times 8 + 5 \times 8^2 + 7 \times 8^3.$$

The number 8 would in this case be called the *radix* of the scale, just as in the common notation 10 is the radix of the scale.

The figures which compose any number are called its *digits*.

The name *binary* is given to a scale if its radix is 2; *ternary*, if 3; *quaternary*, if 4; *quinary*, if 5; *denary* or *decimal*, if 10; *undenary*, if 11; *duodenary* or *duodecimal*, if 12.

152. ***To express a given number in any proposed scale.***

Let N be the given number, and let r be the radix of the scale in which it is to be expressed: also suppose a, b, c... to represent the required digits of the new scale.

Then $N = a + br + cr^2 + dr^3 + \ldots$

Now if we divide N by r, we shall obtain a quotient $b + cr + dr^2 + \ldots$ and a remainder a; hence it is clear that we can find the required digit a by dividing N by r and taking the remainder as the first digit. Again, to determine b, divide the quotient $b + cr + dr^2 + \ldots$ by r, and the new remainder b will be the second digit: in the same way we may continue to divide by r until no quotient is left, and the remainder at each division forms another of the required digits.

Ex. 1. Express 57894 in the scale whose radix is 7.

$$\begin{array}{r|l} 7 & 57894 \\ \hline 7 & 8270 \;.\; 4 \\ \hline 7 & 1181 \;.\; 3 \\ \hline 7 & 168 \;.\; 5 \\ \hline 7 & 24 \;.\; 0 \\ \hline & 3 \;.\; 3 \end{array}$$

Hence the required number is

$4 + 3.7 \times 5.7^2 + 0.7^3 + 3.7^4 + 3.7^5$, or 330534 in the scale 7.

Ex. 2. Transform 46321 from the scale 7 to the scale 9.

$$\begin{array}{r|l} 9 & 46321 \\ \hline 9 & 3554 \;.\; 7 \\ \hline 9 & 366 \;.\; 8 \\ \hline 9 & 22 \;.\; 2 \\ \hline & 1 \;.\; 7 \end{array} \qquad 17287 \text{ Ans.}$$

The student will observe that the division of the above is

performed with reference to the number being in the scale 7: thus in the first line we say 9 will not divide into 4, and then instead of dividing 9 into 46, we multiply 4 by 7 (=28), add 6 (=34), and divide 34 by 9, which gives 3 with remainder 7; we do not then divide 9 into 73, but we multiply 7 by 7 (=49), add 3 (=52), and divide 52 by 9, which gives 5 with remainder 7; next, we multiply 7 by 7, add 2 (=51), and divide 51 by 9, which gives 5 with remainder 6; then we multiply 6 by 7, add 1 (=43), and divide 43 by 9, which gives 4 with remainder 7; and so on in each line.

Ex. 3. Multiply 57012 by 3016 in the scale 8.

57012	In multiplying by 6, we say $6\times2=12$ (in denary scale), i.e. 14 in scale 8; so again $6\times7=42$ (in denary scale)$=52$ in scale 8.
3016	
432074	
57012	
2150360	In adding, we have $7+2=9$ (in denary scale)$=11$ in scale 8.
216260214 Ans.	

153. As in the common scale any digit may occur from 0 to 9, so with numbers in other scales we may have any digit less than the radix. In the scales higher than 10, t is the symbol used for *ten*, and e for *eleven*.

Ex. 4. Extract the square root of $202tt1$ in the scale 12.

	$20'2t't1$ ($4e1$ Ans.	The student will observe that the work here is all done in the scale 12. Thus, as the result of multiplying 4 by itself, we write 14 because *sixteen* is 14 in this scale: so in subtracting we borrow 12 (not 10).
	14	
$8e$	$82t$	
	821	
$9t1$	$9t1$	
	$9t1$	

Ex. 5. In what scale is the number 132500 represented by 90605?

Let r be the radix of the scale; then

$$9r^4+6r^3+5=132500,$$

$$9r^4+6r^3+1=132496,$$

$$3r^3+1=364,$$

$$3r^3=363,$$

$$r=11.$$

Hence the required scale is 11. Ans.

EXERCISE XXIX.

1. Express 57894 in the scale of 7; and 87209 in the scale of 6.

2. Express 32564 in the septenary, and 57340 in the nonary scale.

3. Transform 253 from the common into the binary scale, and 407 into the ternary scale.

4. Transform 46321 from the scale of 7 to that of 9, and bring the result back again to the scale of 7.

5. Transform 8917*t*4 from the scale of 11 to that of 8; and 75403 from the scale of 9 to that of 7.

6. Find the number in the common notation which is the same as 37*te*1 in the scale of 12; and the number in the quinary scale equivalent to 5712 of the octenary.

7. Multiply 34705 by 1402 in the scale of 8; and 154032 by 3021 in the scale of 6.

8. Multiply 75023 by 3*et* in the scale of 12; and 312*t*4 by *t*41 in the scale of 11.

9. Multiply 573102 by 765 in the octenary scale; transform both numbers into the common scale, and transform their product into the octenary scale.

10. Divide 33112563 by 6512 in the scale of 7; and 19224292 by 54*t*7 in the scale of 12.

11. Divide 205124322 by 41205 in the scale 6; and 231007*e*5*t* by 278902 in the scale 12.

12. Divide 17676453 by 7531 in the scale 8; and 1011200120 by 2001 in the scale 3.

13. In what scale is the number 127 represented by 447? and in what is 488 written 965?

14. Find the scale in which 4487 is written 10607.

15. In what scale is 170 the double of 74? and in what scale is 2202102 three times as large as the number 596 of the undenary?

16. In what scale is 1534 of the senary represented by 418?

Find the square root of

17. 42516444 in the scale of 8.

18. *t*2120344 in the scale of 12.

19. 1112001111 in the scale of 3.

20. 36743234 in the scale of 9.

21. *t*67*e*004 in the scale of 12.

22. 14432317 in the scale of 9.

23. 211330221010 in the scale of 4.

24. 13456707 in the scale of 9.

CHAPTER XXX.

PERMUTATIONS AND COMBINATIONS.

154. THE *Permutations* of any quantities are the different orders in which they can be arranged.

Thus if the letters a, b, c are taken two together, six permutations can be formed of them; viz. ab, ba, ac, ca, bc, cb.

155. The *Combinations* of any quantities are the different collections that can be formed out of them without regarding the order in which the quantities are arranged.

Thus of the letters a, b, c taken two together, three combinations can be formed, viz. ab, ac, bc; ab and ba forming the same combination though different permutations.

156. *The number of Permutations of* n *things taken* r *together is* $n(n-1)(n-2)\ldots(n-r+1)$.

Suppose there are n things $a, b, c, d\ldots$; if we remove a there will clearly be $n-1$ things $b, c, d\ldots$, and if we place a before each of these, there will be $n-1$ permutations $ab, ac, ad\ldots$ in all of which a stands first. Similarly we can form $n-1$ permutations $ba, bc, bd\ldots$ in all of which b stands first; so also for c, d;... and since there are n letters each of which stands first in $n-1$ permutations, there will be altogether $n(n-1)$ permutations of n things taken two together.

From this we can determine the number of permutations when they are taken *three* together. If as before we remove a, there will be $n-1$ letters $b, c, d\ldots$ and by

what has just been shewn, they will, if taken two together, form $(n-1)(n-2)$ permutations. Now place a before each of these, and we shall have $(n-1)(n-2)$ permutations of the letters taken three together in all of which a stands first: in the same way we can form $(n-1)(n-2)$ permutations in which b stands first; so also for c, d..... And since there are n letters each of which stands first in $(n-1)(n-2)$ permutations there will be altogether $n(n-1)(n-2)$ permutations of n things taken three together.

We may proceed from this to find the number of permutations when the quantities are taken r together. Suppose the rule which we have found indicated in the case of quantities when taken two or three together to hold good when they are taken $r-1$ together; that is, that the number of permutations of the n quantities a, b, c... taken $r-1$ together is $n(n-1)(n-2)...\{n-(r-1)+1\}$. Let a be removed, and the permutations of the remaining $n-1$ letters b, c, d... taken $r-1$ together will be

$$(n-1)(n-2)(n-3)\ldots\{(n-1)-(r-1)+1\},$$

or
$$(n-1)(n-2)(n-3)\ldots(n-r+1).$$

We can place a before each of these, and we shall have this as the number of permutations of n things taken r together, in all of which a stands first. The same will be true for b, c, d...; hence we see that the permutations of n things taken r together are

$$n(n-1)(n-2)(n-3)\ldots(n-r+1).$$

This result shews us that if this expression be correct when the quantities are taken $r-1$ together, it is also true when they are taken r together, i. e. for the next higher number. But we have proved it to be true when

$r=2$, or 3; hence it is true when $r=4$, and therefore when $r=5$, and so on: that is, it is generally true whatever value r may have.

Ex. 1. In how many ways can a four-oar be arranged, if there are 7 rowers to choose from?

Here there are the permutations of 7 rowers taken 4 together; hence we have

$$7\times6\times5\times4=840. \quad \text{Ans.}$$

157. *The number of Permutations of* n *things taken all together is* $n(n-1)(n-2)\ldots3.2.1$.

In the last Art. let $r=n$, thus the expression becomes

$$n(n-1)(n-2)\ldots(n-n+1)=n(n-1)(n-2)\ldots1.$$

This is frequently written in the form $\lfloor n$ and is read *factorial n*. Thus $\lfloor 5$ stands for $1\times2\times3\times4\times5$.

Ex. 2. In how many ways can 6 persons be arranged at a dinner-table?

$$\text{Ans.}=6\times5\times4\times3\times2\times1=720.$$

158. *The number of Permutations of* n *things in which* a *recurs* p *times,* b *recurs* q *times,* c *recurs* r *times, &c. is*

$$\frac{\lfloor n}{\lfloor p\,.\,\lfloor q\,.\,\lfloor r\ldots}.$$

Let N represent the required number of permutations. Now suppose that in each permutation all the letters a were changed into different letters; since there are p of them, these new letters would form $\lfloor p$ times as many permutations as there were before (Art. 157); i.e. the number of permutations would be $N\times\lfloor p$. Similarly if all the

letters b, c... were changed into different letters, the number would be multiplied by $\lfloor q$, $\lfloor r$...; so that the total number would be $N \times \lfloor p \times \lfloor q \times \lfloor r$.... But now that the recurring letters are all changed into different letters, the number of permutations would be $\lfloor n$; hence

$$N \times \lfloor p \times \lfloor q \times \lfloor r \ldots = \lfloor n ;$$

$$\therefore N = \frac{\lfloor n}{\lfloor p \cdot \lfloor q \cdot \lfloor r \ldots}$$

Ex. 3. How many different words may be made from all the letters of the word *speeches*?

There are 8 letters, in which e occurs thrice, and s twice; hence the permutations $= \frac{1.2.3.4.5.6.7.8}{1.2 \times 1.2.3} = 3360$. Ans.

159. *The number of Combinations of* n *things taken* r *together is*

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

The number of permutations is (Art. 156)

$$n(n-1)(n-2)\ldots(n-r+1),$$

but (by Arts. 155 and 157) each combination of r things will form $r(r-1)(r-2)\ldots1$ permutations; hence the number of combinations is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

By multiplying both numerator and denominator of this fraction by $\lfloor n-r$ we may change it into the form

$$\frac{\lfloor n}{\lfloor r \, \lfloor n-r}.$$

Ex. 4. On how many nights can a different guard, consisting of 4 men, be formed from a company of 10 men?

This is the combination of 10 men taken 4 together; hence we have

$$\frac{10\times9\times8\times7}{1\times2\times3\times4}=210. \quad \text{Ans.}$$

Exercise XXX.

1. Find the number of permutations of 15 things taken 4 together; also of 12 things taken 5 together.

2. Shew that the number of permutations of 8 things taken 5 together is twice as great as those of 16 things taken 3 together.

3. Find the ratio between the permutations of 10 things taken 4 together, and of 7 things taken all together.

4. How many words of four letters may be made from the letters of the word *number*?

5. How many words of five letters may be formed from the letters of the word *Windsor*, and in how many of these will *s* stand first?

6. How many different arrangements may be made for 5 picture-frames when there are 8 pictures to select from?

7. In how many ways can a party of 4 persons be arranged at a table? In how many ways can 7?

8. How many arrangements can be made in a space holding 6 ornamental tiles when there are 12 tiles to choose from?

9. The number of permutations of n things taken 3 together is eight times the number when 2 together: find n.

10. The number of permutations of $2n$ things taken 3 together is ten times that of n things 3 together: find n.

11. The number of permutations of n things 4 together is fifteen times the number of permutations of $n-2$ things 3 together: find n.

12. The number of permutations of $2n$ things taken 3 together : the number of permutations of $n+2$ things taken 3 together :: 14 : 3; find n.

13. How many different words may be made of all the letters of the words *scissors*, *street*, *Zamzummims*, *Ilicifolia*, *assessors*?

14. How many signals may be made with 7 flags when 3 are used at one time? Also how many may be made with 5 flags when any number may be used?

15. With four differently coloured lights how many signals may be given (i) if all are used, (ii) if any number may be used?

16. In a race in which six start, how many different results are possible, excluding dead-heats?

17. Find the number of arrangements that can be made of the letters of *agrarian*. In how many will *a* stand first?

18. Find the number of combinations of 20 things taken 5 together; also of 40 things taken 3 together.

19. Shew that the number of combinations of 12 things taken 3 together is the same as when taken 9 together.

20. If twenty boys enter for a Fives-prize, how many different pairs may be drawn?

21. Out of a stable of 12 horses how many teams of 4 each may be formed, and in how many of them will any particular horse appear?

22. How many different 8-oar crews can be formed out of 15 boys, 3 of whom are steerers?

23. If eggs are 15 for a shilling, how many different shillings worth can be picked out of a basket containing 48 eggs?

24. A house of 11 boys contains 6 who can only row, and 3 who can only steer; the other two can do either. How many different crews can they form for their four-oar?

CHAPTER XXXI.

BINOMIAL THEOREM.

160. IT has been shewn in Chapter V. how a binomial expression may be expanded to any power not greater than the fifth: we shall now shew how this process may be extended to any power, by means of the result generally known as the Binomial Theorem.

161. *To shew that when* n *is any positive integer,* $(x+a)^n$ *is equal to*

$$x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}$$

$$+ \ldots\ldots + \frac{n(n-1)}{1.2}a^{n-2}x^2 + na^{n-1}x + a^n.$$

By actual multiplication

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$

$$(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3$$
$$+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x$$
$$+abcd,$$

in which results we observe the following rules:

(1) That the products proceed in descending powers of x, and that the highest index of x is the same as the number of factors.

(2) That the number of terms is one more than the number of factors.

(3) That the coefficient of the first term is 1; of the second term, the sum of the quantities $a, b, c\ldots$; of the third, the sum of their products taken two together; of the fourth, the sum of their products taken three together; and so on to the last term, which is the product of all the letters.

Now *suppose* the law which we have proved for four factors to hold good for n factors $x+a,\ x+b,\ldots x+l$, so that

$$(x+a)(x+b)(x+c)\ldots(x+k)(x+l)$$
$$=x^n+(a+b+c+\ldots+l)x^{n-1}$$
$$+(ab+ac+\ldots+kl)x^{n-2}+\ldots\ldots+abc\ldots kl$$
$$=x^n+S_1x^{n-1}+S_2x^{n-2}+S_3x^{n-3}+\ldots\ldots+S_n,$$

where
$$S_1=a+b+c+\ldots+l,$$
$$S_2=ab+ac+\ldots+kl,$$
$$S_3=abc+abd+\ldots+hkl,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n=abc\ldots kl.$$

Let us now multiply this result by another factor $x+m$; then

$$(x+a)(x+b)(x+c)\ldots(x+l)(x+m)=x^{n+1}+(S_1+m)x^n$$
$$+(S_2+S_1m)x^{n-1}+(S_3+S_2m)x^{n-2}+\ldots+S_nm.$$

In this, the coefficient of x^n is

$$S_1+m=a+b+c\ldots+l+m$$

=the sum of the letters taken singly:

the coefficient of x^{n-1} is

$$S_2+S_1m=ab+ac+\ldots+kl+am+\ldots+lm$$

=the sum of the letters taken two together:

the coefficient of x^{n-2} is

$$S_3+S_2m=abc+abd+\ldots+abm+acm+\ldots klm$$

=the sum of the letters taken three together:

.....

the last term is $S_nm=abc\ldots klm$=the product of all the letters.

We see therefore that if the result be true for n factors, it follows that it is also true for $n+1$ factors: but we have proved it to be true for three and for four factors, therefore it is true for five; and since for five, therefore also for six; and so on: that is, it is universally true. We have proved therefore that

$$(x+a)(x+b)(x+c)\ldots(x+l)=x^n+(a+b+c+\ldots+l)\,x^{n-1}$$
$$+(ab+ac+\ldots+kl)\,x^{n-2}+(abc+abd+\ldots+hkl)\,x^{n-3}+\ldots$$
$$+abc\ldots kl.$$

Now let $a=b=c=\ldots=l$, and since there are n of them

$$a+b+c+\ldots+l=na,$$

$$ab+ac+\ldots+kl=\frac{n(n-1)}{1.2}a^2 \text{ (Art. 159)},$$

$$abc+abd+\ldots+hkl=\frac{n(n-1)(n-2)}{1.2.3}a^3,$$

.....

$$abc\ldots kl=a^n.$$

Hence we obtain the result

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}$$
$$+\ldots+na^{n-1}x+a^n.$$

162. Ex. 1. Expand $(a+b)^7$.

In this $n=7$; hence

$$(a+b)^7=a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\frac{7.6.5.4}{1.2.3.4}a^3b^4$$
$$+\frac{7.6.5.4.3}{1.2.3.4.5}a^2b^5+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}ab^6+b^7$$
$$=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

Ex. 2. Expand $(x-2y)^6$.

$$(x+2y)^6=x^6-6.x^5.(2y)+\frac{6.5}{1.2}x^4(2y)^2-\frac{6.5.4}{1.2.3}x^3(2y)^3$$
$$+\frac{6.5.4.3}{1.2.3.4}x^2(2y)^4-\frac{6.5.4.3.2}{1.2.3.4.5}x(2y)^5+(2y)^6$$
$$=x^6-12x^5y+60x^4y^2-160x^3y^3+240x^2y^4-192xy^5+64y^6.$$

When the proposed binomial contains factors in one or both of its terms (as in $2y$ in the above), it is better in the expansion to write them first in small brackets, and remove these brackets in obtaining the final result.

163. *To find the general term in the expansion of* $(x+a)^n$.

In the result obtained in Art. 161, we observe that the highest factor of any denominator is always one less than

the number of the term, and the corresponding factor of the numerator is the difference between this factor of the denominator and $n+1$; also that the index of a is the same as the highest factor of the denominator, and the index of x the difference between this factor and n. This shews that the r^{th} term of $(x+a)^n$ is

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)}a^{r-1}x^{n-r+1}.$$

If we multiply both numerator and denominator by $\lfloor n-r+1$, this result may also be written

$$\frac{\lfloor n}{\lfloor r-1 \,.\, \lfloor n-r+1}a^{r-1}x^{n-r+1}.$$

Ex. 3. Find the 6^{th} term in the expansion of $(x-3y^2)^8$.

Here $n=8$, $r=6$:

hence the 6^{th} term

$$=\frac{8.7.6.5.4}{1.2.3.4.5}(-3y^2)^5x^3=56\times(-243y^{10})x^2=-13608x^2y^{10}.$$

164. *The coefficients of any two terms equidistant from the beginning and end of the expansion of* $(x+a)^n$ *are the same.*

Since the first term of the expansion is x^n, and all the inferior powers of x are involved in the remaining terms down to the last term which is a^n, it is evident that there are in all $n+1$ terms: hence the r^{th} term from the end is the $\{n+1-(r-1)\}^{\text{th}}$, i.e. the $(n-r+2)^{\text{th}}$ term from the beginning; and its coefficient is

$$\frac{n(n-1)(n-2)\ldots\{n-(n-r+2)+2\}}{1.2.3.\ldots.(n-r+1)} \quad \text{(Art. 163)},$$

$$=\frac{n(n-1)(n-2)\ldots r}{1.2.3.\ldots.(n-r+1)}$$

$$=\frac{\lfloor n}{\lfloor n-r+1 \,.\, \lfloor r-1}.$$

But this is the coefficient of the r^{th} term of the series (Art. 163), which proves that these two terms have the same coefficient.

Ex. 4. Find the coefficient of the 100^{th} term in the expansion of $(a+x)^{103}$.

Since there are 103 terms, the 100^{th} term is the 4^{th} from the end; its coefficient is therefore the same as that of the 4^{th} term of the series; hence we have

$$\frac{102 \cdot 101 \cdot 100}{1 \cdot 2 \cdot 3} = 171700.$$

165. The proof of the Binomial Theorem in Art. 161 extends only to the case of n being a positive integer: the theorem is however equally true when n is fractional or negative; the student will find the *proof* for these cases in larger works on Algebra, but he need have no difficulty in *applying* the theorem to any case.

Ex. 5. Expand $(1+x)^{\frac{2}{3}}$ to four terms.

$$(1+x)^{\frac{2}{3}} = 1 + \frac{2}{3}x + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1 \cdot 2}x^2 + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1 \cdot 2 \cdot 3}x^3 + \ldots\ldots$$

$$= 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} + \ldots\ldots$$

Ex. 6. Expand $(1-2a)^{-3}$ to five terms.

$$(1-2a)^{-3} = 1 + 6a + \frac{(-3)(-4)}{1 \cdot 2}(-2a)^2 + \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3}(-2a)^3$$

$$+ \frac{(-3)(-4)(-5)(-6)}{1 \cdot 2 \cdot 3 \cdot 4}(-2a)^4 + \ldots\ldots$$

$$= 1 + 6a + 24a^2 + 80a^3 + 240a^4 + \ldots\ldots$$

Ex. 7. Expand $(a^2+3x^2)^{-\frac{3}{4}}$ to four terms.

$$(a^2+3x^2)^{-\frac{3}{4}} = a^{-\frac{3}{2}}\left(1+\frac{3x^2}{a^2}\right)^{-\frac{3}{4}}$$

$$= a^{-\frac{3}{2}}\left\{1-\frac{3}{4}\cdot\frac{3x^2}{a^2}+\frac{\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{1.2}\left(\frac{3x^2}{a^2}\right)^2 + \frac{\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{10}{4}\right)}{1.2.3}\left(\frac{3x^2}{a^2}\right)^3+\ldots\ldots\right\}$$

$$= a^{-\frac{3}{2}}-\frac{9}{4}x^2a^{-\frac{7}{2}}+\frac{189}{32}x^4a^{-\frac{11}{2}}-\frac{1890}{128}x^6a^{-\frac{15}{2}}+\ldots\ldots$$

Exercise XXXI.

Expand

1. $(x+y)^5$.
2. $(x-y)^7$.
3. $(a+b)^6$.
4. $(a-b)^8$.
5. $(1-x)^6$.
6. $(1-a^2)^5$.
7. $(a+2x)^5$.
8. $(x+3y)^6$.
9. $(a^2-x)^8$.
10. $(2-3x)^5$.
11. $\left(2a-\frac{b}{2}\right)^6$.
12. $\left(2-\frac{3x}{4}\right)^5$.

Find the first four terms of

13. $(x-3)^7$.
14. $(x+4y)^{13}$.
15. $(2-5a)^6$.
16. $(x-y)^{20}$.
17. $\left(3+\frac{a}{3}\right)^8$.
18. $(2-3x)^9$.

Expand to five terms

19. $(1-7x)^{\frac{1}{7}}$.
20. $(1-4x^2)^{\frac{3}{2}}$.
21. $(4-3a)^{\frac{5}{2}}$.
22. $(8-3x)^{\frac{4}{3}}$.
23. $(1-3x)^{-1}$.
24. $(1-4x^2)^{-1}$.

25. $(1+a^2)^{-2}$. 26. $(1-2a)^{-3}$. 27. $(1-2x^2)^{-7}$

28. $(a-3x)^{-2}$. 29. $(a+2x)^{-4}$. 30. $\left(x+\frac{a}{3}\right)^{-6}$.

31. $(x^2-a)^{-3}$. 32. $(a^2-2x^2)^{-3}$. 33. $(x^3+2y^3)^{\frac{4}{3}}$.

34. $\left(1-\frac{2x}{3}\right)^{\frac{3}{2}}$. 35. $(1-2x)^{-\frac{1}{3}}$. 36. $(1+3x)^{-\frac{2}{3}}$.

37. $(a-3b)^{-\frac{1}{2}}$. 38. $(a^2+4x^2)^{-\frac{5}{2}}$. 39. $(4a^2-2x^2)^{-\frac{3}{2}}$.

40. $(8a^3-4x)^{-\frac{2}{3}}$.

41. Find the 6th term in the expansion of $\left(1-\frac{2y}{3}\right)^{-1}$.

42. Determine the coefficient of x^8 in $(1-3x^2)^{\frac{1}{4}}$; also of x^{10} in $(1-2x^3)^{-\frac{1}{2}}$; also of z^{25} in $(1+z^5)^{-\frac{2}{3}}$.

CHAPTER XXXII.

Proofs of Rules.

166. We shall give in this chapter the *proofs* of some of the elementary rules which the student would probably have found too difficult at his first introduction to Algebra.

167. *To prove that* $(a+b)(c+d)=ac+ad+bc+bd$.

The product of a and c being written ac, it follows from Arithmetic that

$$(a+b)c=ac+bc \quad \text{.....(i),}$$

because $(a+b)c$, i.e. $(a+b)$ times c is equal to

$$(a \text{ times } c)+(b \text{ times } c),$$

which is written $ac+bc$.

From this we can determine the value of $(a+b)(c+d)$: for let $c+d=x$; then

$$\begin{aligned}(a+b)(c+d)&=(a+b)x,\\&=ax+bx,\\&=a(c+d)+b(c+d),\\&=ac+ad+bc+bd \text{ by (i).}\end{aligned}$$

Similarly to find the value of (a−b)(c−d).

We know that $(a-b)c=ac-bc$,

because $(a-b)c$ is the product of a and c (i.e. ac) diminished by the product of b and c (i.e. bc), which is written $ac-bc$. Now let $c-d=x$, and we have

$$\begin{aligned}(a-b)(c-d)&=(a-b)x,\\&=ax-bx,\\&=a(c-d)-b(c-d),\\&=ac-ad-bc+bd.\end{aligned}$$

168. *To state and prove the Rule for finding the Greatest Common Measure of two Algebraical quantities.*

Let the expressions be arranged according to ascending or descending powers of some common letter; then divide one by the other, and as in Arithmetic take the remainder for a new divisor, and the old divisor for a new dividend; continue this process until there is no remainder; and the last divisor will be the G.C.M.

The proof of the Rule depends upon the following principles:

(i) If one quantity measure another, it will also measure any multiple of that quantity. For let A be a measure of B and contained k times in it; then $B=kA$;

$$\therefore\ hB=hkA,$$

hence A is contained hk times in hB, or A is a measure of any multiple of B.

(ii) If a quantity be a measure of two others, it will also be a measure of the sum or difference of any of their multiples.

Let M be contained h times in A, and k times in B; then $A=hM$, and $B=kM$;

$$\therefore pA \pm qB = phM \pm qkM = (ph \pm qk)M,$$

i.e. M is a measure of $pA \pm qB$.

Now let A and B represent two quantities, and let B be contained p times in A with a remainder C; again let C be contained q times in B with a remainder D; and so on until there is no remainder; let D be the last divisor; then it will be the G.C.M. of A and B.

$$\begin{array}{r|l} B & A\,(p \\ & \underline{pB} \\ C & B\,(q \\ & \underline{qC} \\ D & C\,(r \\ & \underline{rD} \\ & 0 \end{array}$$

From the above division we have the following relations:

$$A - pB = C \quad \text{..... (i)},$$

$$B - qC = D \quad \text{..... (ii)},$$

$$C = rD \quad \text{..... (iii)}.$$

From (iii) it is evident that D is a measure of C; therefore by (ii) it is a measure of B, and by (i) it is also a measure of A: hence D is a common measure of A and B. But it is also their greatest common measure, for by (i) every common measure of A and B is also a measure of C, i.e. it is a common measure of B and C; therefore by (ii) it is also a measure of D, i.e. every common measure of A and B divides D: but we have shewn that D is a common measure of A and B, and no quantity larger than

D can divide D; hence D is the greatest common measure of A and B.

169. *To state and prove the Rule for finding the Least Common Multiple of two Algebraical quantities.*

Find the G.C.M. of the two quantities, and divide one of them by it; the L.C.M. will be the product of this quotient and the other quantity.

Let A and B be the two quantities, D their greatest common measure, M their least common multiple; also let $A=pD$, $B=qD$. Then D being the *greatest* common measure of A and B, the least quantity which will contain A and B is pqD, for D includes all factors common to A and B, p all those peculiar to A, and q all those peculiar to B;

$$\therefore\ M=pqD=\frac{A}{D}\times\frac{B}{D}\times D=\frac{A}{D}\times B.$$

It is evident that the L.C.M. may also be written in the form of three products, viz. the G.C.M. and the quotient of each of the quantities when divided by the G.C.M.

170. *To prove that*

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1}.$$

By actual division

$$\frac{x^2-a^2}{x-a}=x+a,$$

$$\frac{x^3-a^3}{x-a}=x^2+xa+a^2,$$

$$\frac{x^4-a^4}{x-a}=x^3+x^2a+xa^2+a^3.$$

Now *suppose* this law which we see is true in these simple cases to hold good when the index in the numerator is n, so that

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1};\ldots\ldots(\mathrm{A})$$

then it follows that

$$\begin{aligned}\frac{x^{n+1}-a^{n+1}}{x-a}&=x^n+\frac{x^na-a^{n+1}}{x-a}\\&=x^n+a\cdot\frac{x^n-a^n}{x-a}\\&=x^n+a\,(x^{n-1}+x^{n-2}a+\ldots+a^{n-1})\text{ by (A)}\\&=x^n+x^{n-1}a+x^{n-2}a^2+\ldots a^n.\end{aligned}$$

But this is of the same form as the expression (A); we find therefore that supposing the rule to hold for any index n, it necessarily holds for the next highest index $n+1$. But we have shewn that it is true for indices 2, 3, 4; therefore it is true for 5, and therefore for 6, and so on: that is, it is generally true.

The same method of inductive proof may be applied to all the cases stated in Art. 35.

Exercise XXXII.

Miscellaneous Examples.

1. If $a=0$, $b=2$, $c=4$, $d=6$, find the value of

$$3\sqrt[3]{2b^2-a}+2\sqrt[3]{b^4+c^2+7}-\sqrt[3]{2(b+c)^2-(b+d)^2}.$$

2. Divide $49a^2+21ab+12bc-16c^2$ by $7a+3b-4c$.

3. Resolve the following expressions into factors:

$$x^2+x-132;\qquad x^3+125;\qquad x^4-81.$$

4. Find the L.C.M. of $6x^3+x-2$, and

$$9x^3+48x^2+52x+16.$$

5. Simplify the following expressions:

(i) $\dfrac{x^4+a^2x^2+a^4}{x^6-a^6}$.

(ii) $\dfrac{2a}{a-b}-\dfrac{2a}{a+b}+\dfrac{4b^2}{a^2+b^2}$.

(iii) $\dfrac{x-y}{x^2-xy+y^2}+\dfrac{1}{x+y}+\dfrac{xy}{x^3+y^3}$.

(iv) $\left(x+\dfrac{xy}{x-y}\right)\times\left(y-\dfrac{xy}{x+y}\right)$.

6. If $(by-cx)^2=(b^2-ac)(y^2-cz)$, prove that

$$(bx-ay)^2=(b^2-ac)(x^2-az).$$

7. Solve the equations:

$$\left.\begin{aligned}10x-2y+4z&=10,\\3x+5y+3z&=20,\\x+3y-2z&=21.\end{aligned}\right\}$$

8. Solve

(1) $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=7.$

(2) $x-y=2$ and $x^3-y^3=218.$

9. A man buys a case of oranges at 6*d.* a dozen: he finds fifty spoiled, and selling the rest at 3 for 2*d.* makes 10*s.* 9*d.* profit: how many are there in the case?

10. Extract the cube root of ·007077888.

11. If $a=0$, $b=1$, $c=3$, $d=4$, find the value of

$$(a+c)^b+a^b+c^d+d^{\frac{b+c}{2}}+abcd.$$

12. Divide

$a^4+b^4-c^4+4ab(a^2+b^2)+6a^2b^2$ by $a^2+b^2+c^2+2ab$.

13. Find the factors of

x^5+y^5; x^2-x-12; $6x^3+7x^2-3x$.

14. Find the G.C.M. and the L.C.M. of

x^4-y^4 and x^6-y^6.

15. Simplify

$$\frac{a+b}{x+a}+\frac{a-b}{x-a}-\frac{2a(x+b)}{x^2+a^2}.$$

16. Extract the square root of

$$x^4-2x^3+5x^2-6x+8-\frac{6}{x}+\frac{5}{x^2}-\frac{2}{x^3}+\frac{1}{x^4}.$$

17. Solve the following equations:

(i) $2x^2+3x+5=3x^2+4x-1$.

(ii) $\dfrac{x+2}{x-4}+\dfrac{2x+11}{x-2}=14$.

(iii) $2x^2=3xy$, $y^2+5xy=34$.

18. Find the time between 8 and 9 o'clock when the hour and minute-hands of a clock are at right angles.

19. Multiply

$a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{-\frac{2}{3}}-a^{-\frac{1}{3}}b^{-\frac{1}{3}}+b^{-\frac{2}{3}}$.

20. Simplify

$$\left(a+\frac{b-a}{1+ba}\right)\times\frac{a}{b}\div\left(1-a\frac{b-a}{1+ba}\right),$$

and find the value of

$$\frac{x^5+ax^4-a^4x+a^5}{x^4+2ax^3+2a^2x^2-2a^3x+a^4}$$ when $x=a$.

21. Find the square root of

$$16x^4 - \tfrac{8}{3}x^3y - \tfrac{7}{9}x^2y^2 + \tfrac{2}{27}xy^3 + \tfrac{1}{81}y^4.$$

22. Find the G.C.M. of

$$3x^3 - 5x + 2, \text{ and } 4x^3 - x - 4x^2 + 1.$$

23. Simplify $\dfrac{a}{a-b} + \dfrac{a}{a+b} + \dfrac{2a^2}{a^2+b^2} + \dfrac{4a^2b^2}{a^4-b^4}$.

24. Solve the following equations:

(i) $\dfrac{10x+3}{3} - \dfrac{6x-7}{2} = 10(x-1)$.

(ii) $3x - 2y = 6$; $3y - 2z = 5$; $3z - 2x = -2$.

(iii) $5x^2 + 7x - 2 = x^2$.

(iv) $x + y = 11$; $x^3 + y^3 = 737$.

25. A person gives away £10 less than $\frac{1}{3}$ of his money and has left £20 more than two-fifths of it. Find the original sum.

26. Form the equation whose roots are 5, 0, and $\frac{1}{2}$.

27. Divide $3\sqrt{32}$ by $4\sqrt{18}$;

and $a^3b^{-\frac{3}{4}} + a^{-3}b^{\frac{3}{4}}$ by $ab^{-\frac{1}{4}} + a^{-1}b^{\frac{1}{4}}$.

28. Find the square root of $12 - 6\sqrt{3}$.

29. What number must be added to each term of the ratio 10 : 15, that the new ratio may be as 24 to 28?

30. If a, b, c, d are in proportion shew that

$$c^2 + bd : b^2 + ac :: d : b.$$

31. Divide $a^{-4} + a^{-2}b^{-2} + b^{-4}$ by $a^{-2} - a^{-1}b^{-1} + b^{-2}$.

32. Solve the equations:

(i) $\left.\begin{aligned} \frac{4x+5y}{40} &= x-y, \\ \frac{2x-y}{3}+2y &= \tfrac{1}{2}. \end{aligned}\right\}$

(ii) $x^3-y^3=56, \qquad x-y=2.$

33. Of a certain sum of money I spent a third part and put in its stead £50; then of the sum thus increased I spent one-fourth, and put in its stead £70; I then had £120; what was the original sum?

34. If $2s=a+b+c$ and $A=\frac{b^2+c^2-a^2}{2bc}$,

shew that $1-A=\frac{2(s-b)(s-c)}{bc}$.

35. Find the value of

$pq-(p^2-1)^{\frac{1}{2}}(q^2-1)^{\frac{1}{2}}$ when $2p=x+\frac{1}{x}$, $2q=y+\frac{1}{y}$.

36. Solve the equations:

(i) $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{1}{f}(fh-ex)$.

(ii) $2x^2-17x=30$.

37. Simplify $\left(\frac{a-b}{a+b}+\frac{a+b}{a-b}\right)\div\left(\frac{a^2-b^2}{a^2+b^2}+\frac{a^2+b^2}{a^2-b^2}\right)$.

38. Find five numbers in Arithmetical progression, such that their sum is 55, and the sum of their squares 765.

39. Insert m geometric means between a and b.

40. Divide 23165413 by 2403 in the scale 8.

41. Simplify $4x^3-\{2x^2-[5x^3-3x^2-(2x+7)-4]-9x^3\}$.

42. Reduce to lowest terms $\dfrac{x^3-4x^2-19x-14}{2x^3-9x^2-38x+21}$.

43. Solve the equations:

(i) $5x-3y+2z=21$, $8x-y-3z=3$, $2x+3y+2z=39$.

(ii) $3x^2-26x=169$.

44. Divide $x^2-4a^{\frac{3}{4}}x^{\frac{1}{4}}+4a^{\frac{1}{4}}x^{\frac{3}{4}}-a^2$ by $x^{\frac{3}{4}}-a^{\frac{3}{4}}$.

45. Two numbers are in the ratio of 7 : 8; and if 9 be added to each of them, they will be in the ratio of 10 : 11; find the numbers.

46. A is three times as old as B, and 19 years hence he will be only twice as old as B: what is A's age?

47. Sum the following series:

(i) $1,\ 2\frac{2}{3},\ 4\frac{1}{3}\ldots$ to 9 terms.

(ii) $\frac{2}{3},\ \frac{5}{6},\ 1\ldots$ to n terms.

(iii) $1,\ \frac{1}{4},\ \frac{1}{16}\ldots$ to infinity.

48. How many combinations can be formed of the letters in the word *longitude* taken 4 at a time?

49. Expand $(a-2x)^8$ to 4 terms.

50. If α, β, are the roots of the equation $ax^2+bx+c=0$, shew that $\alpha\beta=\dfrac{c}{a}$, $\alpha+\beta=-\dfrac{b}{a}$.

For what value of k will the equation $5x^2+6x+k=0$ have equal roots?

51. Simplify $\dfrac{\frac{1}{a}+\frac{1}{ab^3}}{b-1+\frac{1}{b}}$.

52. Divide $a+b$ by $\sqrt[3]{a}+\sqrt[3]{b}$.

53. Extract the square root of $28+16\sqrt{3}$.

54. Solve the equation,

$$\frac{1}{\sqrt{1+x}+1}+\frac{1}{\sqrt{1+x}-1}=\frac{1}{x}.$$

55. Find how many terms of the series 7, 9, 11, &c., amount to 72.

56. Insert 3 Arithmetic Means between 117 and 477.

57. Sum the series $9-6+4$..... to 8 terms, and to infinity.

58. Insert 4 Harmonic Means between 2 and 12.

59. If $a : b :: c : d$, shew that

$$7a+b : 3a+5b :: 7c+d : 3c+5d.$$

60. If I buy a horse and sell it again for £96 I gain as much per cent. as the horse cost me. Find the original cost of the horse.

61. Find the G. C. M. and L. C. M. of

$$4x^4-6x^3y+18xy^3+8y^4$$

and

$$10x^4+17x^3y-11xy^3-4y^4.$$

62. If ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor, then $(ac'-a'c)^2=(ab'-a'b)(bc'-b'c)$.

63. Solve the equations:

(1) $\dfrac{x-5}{x-6}+\dfrac{x-11}{x-12}=\dfrac{2x-4}{x+2}$.

(2) $\sqrt{2x+5}+\sqrt{x-1}=\sqrt{x-7}+\sqrt{2x+11}$.

(3) $x^2+3y^2=52$; $4y^2+3xy=88$.

64. Find the cube root of 29503629.

65. Simplify $\dfrac{2\sqrt{2}-\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$ and $\sqrt{21+8\sqrt{5}}$.

66. Find two numbers in the ratio of 3 to 5, which, when each is reduced by 7, are in the duplicate ratio of 2 to 3.

67. Shew how to find the sum to infinity of

$$2a+a+\frac{a}{2}+\ldots\ldots$$

68. Express 708 in the scale of 9, and multiply the result by itself.

69. Expand $(1-x)^7$ and give the first 4 terms of the expansion of $(3a-2x)^{-\frac{1}{3}}$.

70. A man starting at 10.30 A.M. rows $10\frac{1}{2}$ miles up a river, and rests for 1 hour and 40 mins.; coming back he travels 1 mile per hour faster than before, and reaches home at 5.30 P.M.: what was his rate at first?

71. Simplify

$$\frac{6a^4-2b^2x^2}{a^4+x^4}-\frac{3a^2-b^2}{a^2+x^2}-\frac{3a^2+b^2}{a^2-x^2}.$$

72. Write down the square of

$$1+3x+5x^2+7x^3-9x^4.$$

73. Find two numbers, such that half the second together with twice the first is equal to $13\frac{1}{2}$, and twice the second together with a fifth of the first is equal to 15.

74. Find the square root of $79+24\sqrt{7}$.

75. Divide $x^{\frac{1}{2}}-2x^{\frac{1}{4}}+1$ by $x^{\frac{1}{6}}-2x^{\frac{1}{12}}+1$.

76. Find the sum of the series

$$3\tfrac{1}{2}+4\tfrac{5}{6}+6\tfrac{1}{6}+\text{\&c. to 19 terms.}$$

77. Solve

$$x^2+2xy+3y^2=19;\ 3x^2+2xy+y^2=9.$$

78. Multiply $tt9$ by $10t$ in the undenary scale.

79. Extract the square root of 10534521 in the senary scale.

80. Expand $(a-2x)^6$.

81. Find the G. C. M. and L. C. M. of

$$2x^2+x-1,\ x^2+5x+4, \text{ and } x^3+1.$$

82. Find the value of

$$\frac{x-a}{x-b}+\frac{x-b}{x-a}-\frac{(a-b)^2}{(x-a)(x-b)}.$$

83. Extract the square root of 1 and the cube root of 132651.

84. Solve the equations:

(i) $$\left.\begin{aligned} x+y+z&=36,\\ x+3y-2z&=12,\\ x+\frac{y}{3}+\frac{z}{2}&=20. \end{aligned}\right\}$$

(ii) $4x^2+3xy=10;\ 2xy+3y^2=16.$

85. Expand $(1-a^2)^{\frac{1}{3}}$ to five terms.

86. Expand $(1+x^2)^{-\frac{1}{2}}$ to five terms.

87. Find the 5th term of $(a^2-b^2)^{12}$.

88. Transform 87432 from the nonary to the undenary scale.

89. Divide 833318 by 2471 in the duodenary scale.

90. How many words, each containing 3 consonants and 2 vowels, can be made out of an alphabet of 17 consonants and 5 vowels? In how many of them will the consonants and vowels be arranged alternately?

91. Solve the equations:

(i) $\dfrac{x-1}{x+2}=\dfrac{2x+1}{5x-2}$.

(ii) $\left.\begin{aligned} x(x+y)+y(x-y)&=158;\\ 7x(x+y)&=72y(x-y). \end{aligned}\right\}$

92. If two numbers differ by n, shew that the difference of their squares is n times their sum.

93. Find the value of

$$\frac{a+c}{(a-b)(x-a)}-\frac{b+c}{(a-b)(x-b)}.$$

94. Solve the equations:

$$x+2y+3z=17;\ 2x-3y+z=0;\ 3x+y-5z=-15.$$

95. Find the G.C.M. of

$$2x^4+11x^3-13x^2-99x-45 \text{ and } 2x^3-7x^2-46x-21.$$

96. Divide $m-4m^{\frac{3}{4}}+6m^{\frac{1}{2}}-4m^{\frac{1}{4}}+1$ by $m^{\frac{1}{2}}-2m^{\frac{1}{4}}+1$; and find the value of $(a^{\frac{1}{3}}+b^{\frac{1}{2}})(a^{-\frac{1}{3}}-b^{-\frac{1}{2}})$.

97. If a, b, c, be in continued proportion, prove that

$$a+b : b :: a-c : b-c.$$

98. Sum the series -12, -8, -4... to 14 terms; also $\frac{3}{2}+1+\frac{2}{3}+\ldots$ to 6 terms, and to infinity.

99. What number must be added to each term of the ratio 5 : 13 that it may become 1 : 2?

100. Transform 5279 from the denary scale to the senary; and find in what scale the number 136 is expressed by 253.

101. If $a=6$, $b=4$, $c=2$, $d=0$, find the value of

$$2\sqrt{a-c}+3\sqrt[3]{3a+2b+1}-\sqrt{\frac{bd+c+2b+a}{2c^2+2b^2-a^2-3}}.$$

102. Reduce to its lowest terms

$$\frac{15x^3+35x^2+3x+7}{27x^4+63x^2-12x^3-28x}.$$

103. Resolve into elementary factors

$$x^3-8y^3,\ a^6-2a^4b^2+a^2b^4,\ (4a+3b)^2-(3a+4b)^2.$$

104. Solve the equations:

(i) $x+2y=7$; $y+2z=2$; $3x+2y-z=-5$.

(ii) $2x^2-3x=54$.

105. Find the value of

$$\frac{1}{2(a-x)}+\frac{1}{2(a+x)}+\frac{a}{a^2+x^2}.$$

106. Find the square root of $44\frac{1}{9}\frac{9}{7}\frac{3}{1}\frac{4}{1}$.

107. Prove that

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}=\frac{1}{abc}.$$

108. Rationalise the denominator of

$$\frac{3-\sqrt{2}}{3+\sqrt{2}}.$$

109. Reduce $\frac{3}{5}\sqrt[3]{5}$ to an entire surd; and simplify

$$\sqrt{27}+2\sqrt{48}+3\sqrt{108}.$$

110. Extract the square root of $18+2\sqrt{77}$.

111. Divide

$$x^6-140x^4+1050x^3-3101x^2+3990x-1800$$

by $x^3-12x^2+47x-60$.

112. *A* spends a sum of money; *B* spends half of what *A* spends; *C* spends $\frac{3}{4}$ of what *A* and *B* together spend. If *A*, *B*, *C* altogether spend £52. 10*s*., what does *C* spend?

113. A bag contains sixpences, shillings, and half-crowns; the three sums of money expressed by the different coins are the same; if there are 102 coins in the bag, find the number of sixpences, shillings, and half-crowns.

114. Reduce

$$\frac{x^4-4x^3+8x^2-8x-21}{x^4-x^3+12x^2-7x+49}$$

to its lowest terms.

115. Solve the equation

$$\frac{x+1}{6}-\frac{2x-6}{7}=6-\frac{4x+13\frac{2}{3}}{14}.$$

116. There are three numbers in A.P. whose sum is 10, and the product of the second and the third is $33\frac{1}{3}$: find the terms.

117. Sum to 5 terms and to infinity the geometrical series $22\frac{1}{2}$, 15, 10, &c.

118. Find the value of $(1-3x+3x^2-x^3)^3$.

119. Extract the cube root of

$$8x^6-36x^5+66x^4-63x^3+33x^2-9x+1.$$

120. Multiply together 325 and 104 which are in the scale of 6: and express the result in the common scale.

121. Simplify

(1) $\dfrac{1}{xy-y^2}-\dfrac{1}{xy+y^2}-\dfrac{2}{x^2-y^2}$.

(2) $\left\{1+\dfrac{a-x}{a+x}\right\}\div\left\{1+\dfrac{a^2+x^2}{a^2-x^2}\right\}$.

(3) $\dfrac{1}{2x^2-4x+2}+\dfrac{1}{2x^2+6x+4}-\dfrac{1}{1-x^2}$.

122. Solve the equations:

(1) $7x^2-11x=6$.

(2) $(a+b+x)^2-2bx=a^2+2b^2$.

123. Solve the equations:

(1) $2\sqrt{x^2-8x+8}=(x-4)^2-7$.

(2) $2\sqrt{4x+1}=\sqrt{2x+5}+3$.

124. Rationalise the denominator of

$$\frac{\sqrt{20}-\sqrt{12}}{2\sqrt{3}-\sqrt{5}};$$

and extract the square root of $8-2\sqrt{15}$.

125. If α, β are the roots of the equation $ax^2-bx+c=0$, express $\alpha^3+\beta^3$ in terms of a, b and c.

126. Expand $\left(x-\frac{x^2}{2}\right)^{-1}$ to five terms.

127. Find in what scale the number 124 is represented by 147, and in what scale 147 is represented by 124.

128. Transform $3t97e$ from the duodenary to the octenary scale.

129. The 4th term of a series in Arithmetical Progression is $2\frac{1}{2}$, the 9th is $3\frac{1}{2}$; find the 26th term.

130. Out of a certain sum of money A takes £10, and $\frac{1}{3}$ of the remainder, B takes £10 and $\frac{1}{2}$ the new remainder; C takes the rest which is £25. What is the whole sum?

131. Divide $x^3-(a+p)x^2+(q+ap)x-aq$ by $x-a$.

132. Resolve $(x-a)^5-x^5+a^5$ into its factors.

133. Divide $x^{10}+\sqrt{3}x^5+1$ by $x^2-\sqrt{3}x+1$.

134. Find the highest common divisor of
$x^3-2x^2+3x-18$ and $x^3+7x^2-11x-57$.

135. Solve the equations
$a^2(y+z)=b^2(z+x)=c^2(x+y)=xyz$.

136. Shew that $\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}\ldots$ ad infinitum $=a$.

137. The number of permutations of $2n+1$ things taken $n-1$ together : number of permutations of $2n-1$ things taken n together :: 3 : 5. Find the value of n.

138. Extract the square root of $7+\sqrt{13}$; and divide $5-2\sqrt{3}$ by $5+2\sqrt{3}$.

139. If the square of x vary as the cube of y, and $x=2$ when $y=3$, find the relation between x and y.

140. Expand $(1-2x+3x^2-2x^3+x^4)^2$ and $(a^2-ab+b^2)^3$.

141. Shew that

$$a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(b-c)(c-a)(a-b).$$

142. If $m=(x+2)(x-1)^2$, and $n=(x-2)(x+1)^2$, prove that $\dfrac{m-n}{m+n}=\dfrac{2}{x(x^2-3)}$.

143. Simplify

(i) $\dfrac{(a^2-b^2)(c+d)}{(c^2-d^2)(a+b)}$, (ii) $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$,

(iii) $\dfrac{\dfrac{x^2+1}{2x-1}-\dfrac{1}{2}x}{\dfrac{2+x}{1-2x}}$.

144. Find the G.C.M. of $175a^3-210a^2x+63ax^2$ and $125a^3-225a^2x+135ax^2-27x^3$;
also the L.C.M. of $(6-4x)(3-6x)$, $(9-6x)(8-12x)$, and $(6-12x)(6-9x)$.

145. Shew that

$$2\sqrt{84}-4\sqrt{18}+\sqrt{189}+5\sqrt{72}+\sqrt{32}-2\sqrt{242}=7\sqrt{21}.$$

146. Extract the square root of $x^2y^{-2}+y^2x^{-2}-2$ and the cube root of 8741816.

147. Solve the equations:

(i) $\dfrac{1}{20}\{(2x+3)^2-1\}-\dfrac{1}{5}(x^2+6)=-\dfrac{1}{5}$.

(ii) $\dfrac{x}{a}-\dfrac{x-a}{b}=\dfrac{b}{a}$. (iii) $\sqrt{x+a}+\sqrt{x-a}=\sqrt{b}$.

(iv) $2x-5+\dfrac{1}{2x-5}=3\frac{1}{3}$.

148. Expand to four terms $(1-x)^{-4}$ and $(1+3x)^{-2}$.

149. A man sells a horse by lottery; if the tickets were 5*s.* each, he would lose £5; if they were 6*s.* each he would gain £4. Find the price and the number of tickets.

150. A man bought a number of articles for £90. If he had bought 3 more for the same sum, the average price would have been £1. 10*s.* less than it was. How many did he buy?

151. What is the L.C.M. of

$$abxy-b^2y^2,\ a^2x^2-b^2y^2,\ a^2x^2+abxy?$$

152. Find the value of $\dfrac{a+x}{a^2-ax}-\dfrac{3a+x}{a^2-x^2}-\dfrac{a-x}{ax+x^2}$.

153. Extract the cube root of

$$8-36ax+102a^2x^2-171a^3x^3+204a^4x^4-144a^5x^5+64a^6x^6.$$

154. A certain number of 2 digits is less by 4 than 5 times the number obtained by inverting the digits, and the sum of the digits is 10. What is the number?

155. Multiply $3\sqrt{2}-\sqrt{15}$ by $2\sqrt{3}+\sqrt{10}$.

156. Simplify $\dfrac{\{(a^m)^{\frac{1}{r}}(a^q)^{\frac{1}{m}}\}^{mr}}{\{\sqrt[q]{b^m}(\sqrt[m]{b})^r\}^{mq}} \div \left\{\left(\dfrac{a}{b}\right)^q\right\}^r$.

157. Solve the equations:

(i) $x^2+\dfrac{1}{x^2}-a^2-\dfrac{1}{a^2}=0$.

(ii) $x^2-xy+y^2=7,\ x^4+x^2y^2+y^4=133$.

158. Prove that if $\dfrac{a_1+a_2x}{a_2+a_3y}=\dfrac{a_2+a_3x}{a_3+a_1y}=\dfrac{a_3+a_1x}{a_1+a_2y}$, each of these ratios is equal to $\dfrac{1+x}{1+y}$, supposing $a_1+a_2+a_3$ not to be zero.

159. If z is the harmonical mean between a and b,

$$\frac{1}{z-a}+\frac{1}{z-b}=\frac{1}{a}+\frac{1}{b}.$$

160. How many words may be made out of all the letters of the word *College*? How many of *Woolloomooloo*?

161. Solve the equations:

(1) $\dfrac{x-2a}{a-b}-\dfrac{x+2a}{a+b}=\dfrac{2ax}{a^2-b^2}$.

(2) $\sqrt{16-x}+\sqrt{2x-5}=6$.

(3) $\sqrt[3]{x^2}-13\sqrt[3]{x}+36=0$.

(4) $\sqrt{x^2-6x+17}+2x^2=12x-13$.

162. Multiply $x^{\frac{1}{2}}-2y^{\frac{1}{3}}+4x^{-\frac{1}{2}}y^{\frac{1}{3}}$ by $x^{\frac{1}{2}}+2y^{\frac{1}{3}}+4x^{-\frac{1}{2}}y^{\frac{1}{3}}$.

163. Simplify $\sqrt{48}+2\sqrt{27}-\sqrt{75}+3\sqrt{12}$.

164. Simplify $\dfrac{3\sqrt{6}+2\sqrt{2}}{3\sqrt{6}-2\sqrt{2}}+\dfrac{3\sqrt{6}-2\sqrt{2}}{3\sqrt{6}+2\sqrt{2}}$.

165. Extract the square root of $17-\sqrt{253}$.

166. Sum the series

(1) 3, 8, 13, 18, &c. to 100 terms.

(2) 4, 12, 36, &c., to 6 terms.

(3) $2\frac{1}{3}$, $\frac{7}{9}$, $\frac{7}{27}$, $\frac{7}{81}$, &c. to infinity.

167. How many terms of the Arithmetic series 3, 5, 7, 9, &c. make 1680?

168. Find the price of a dozen apples, when two more in a shilling's worth would reduce the price by a penny a dozen.

169. If the number of combinations of $n+2$ things 4 together be equal to the number of combinations of n things 2 together, find n.

170. Prove that the difference of two quantities divided by the difference of their reciprocals is equal to $\pm$ (the product of the quantities).

171. Simplify $a^2+[b(a+b)-\{ab+a(a-b)\}]$,

and find the value of $x^2+2x^{-1}-\sqrt{x}$ when $x=1$.

172. Shew that

$$a(a-b+c)(a+b-c)-c(-a+b+c)(a-b+c) = (a-c)\{(a+c)^2-b^2\}.$$

173. Resolve x^4+3x^2+6x and $x^4+x^2y^2+y^4$ into factors.

174. Reduce to their simplest forms

$$\frac{x}{2}\left(\frac{1}{x-y}-\frac{1}{x+y}\right)\frac{x^2-y^2}{x^2y+xy^2}, \text{ and } \frac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}.$$

175. Solve the equations:

(i) $x^2+6x=55$; (ii) $\sqrt{x+3}+\sqrt{2x-1}=9$;

(iii) $\left.\begin{aligned} x^2+xy&=104,\\ xy-y^2&=15. \end{aligned}\right\}$

176. The terms of a ratio are seven and three; what number must be added to each in order that the ratio may be halved?

177. If $$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3},$$

prove that $$\frac{c_1a_2a_3+c_2a_3a_1+c_3a_1a_2}{c_1b_2b_3+c_2b_3b_1+c_3b_1b_2}=\frac{a_1^2}{b_1^2}.$$

178. Write down the first four terms of an A.P. in which the first term is 3, and the 40th term is 6.

179. Sum to 12 terms the series $\frac{5}{4}-\frac{5}{2}+5-10+\ldots$

180. If $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right)=\frac{2x}{a}$,

shew that

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\right)\left\{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\right)^2-3\right\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

181. Resolve into factors $x^2+6x-187$; and

$$\{(a+b)^2-(c+d)^2\}^2-\{(a-b)^2+(c-d)^2\}^2.$$

182. Simplify

$$\frac{x^2-4}{x^2+4x+3}\times\frac{x^2+6x+9}{x^2+x-6}\times\frac{x^2+5x+6}{x^2-9}.$$

183. Distinguish between $(x^m)^n$ and x^{m^n}; and divide

$$(x^{\frac{1}{2}})^5-(y^{\frac{1}{2}})^5 \text{ by } (x^5)^{\frac{1}{2}}-(y^5)^{\frac{1}{2}}.$$

184. If $x=\sqrt{3}-1$ and $y=2+\sqrt{3}$, find the value of

$$\frac{x-y}{x+y}+\frac{x+y}{x-y}.$$

185. Solve the equations:

(i) $(ax+c)(bx+d)=(ax+e)(bx+f)$.

(ii) $\frac{3}{x}+\frac{4}{y}=17$; $\frac{4}{x}-\frac{1}{y}=10$.

(iii) $(2x-5)(3x-2)-(x-2)(2x-3)=4$.

(iv) $x^2-3x+3-\sqrt{3x^2-7x+6}=\frac{x-x^2}{2}$.

186. Find the sum of the series 3, $2\frac{1}{2}$, 2, $1\frac{1}{2}$,... to 20 terms.

187. If $4a+3b \propto 7a+4b+3$, and when $a=3$, $b=3$, shew that $a=8b-21$.

188. If α, β be the roots of $x^2-px+q=0$, shew that the equation having $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$ for its roots is $q(x+1)^2=p^2x$.

189. If $a+d=2c$, and $\frac{1}{a}+\frac{1}{d}=\frac{2}{b}$, shew that

$$a : b :: c : d.$$

190. It is between two and three o'clock, and in ten minutes the minute-hand will be as much before the hour-hand as it is now behind it. What is the time?

191. Divide $x^{-\frac{1}{2}}-y^2$ by $x^{-\frac{1}{2}}-y^{\frac{1}{2}}$; and determine when $x^n \pm y^n$ is divisible by $x \pm y$.

192. Simplify

$$\frac{a(b+c-a)}{(a-b)(a-c)}+\frac{b(c+a-b)}{(b-c)(b-a)}+\frac{c(a+b-c)}{(c-a)(c-b)}.$$

193. If α, β be the roots of the equation

$$ax^2+2bx+c=0,$$

prove that $ax^2+2bx+c=a(x-\alpha)(x-\beta)$.

If $ax^2+2bx+c=0$, $a'x^2+2b'x+c'=0$ have a common root, and $\frac{a}{a'}$, $\frac{b}{b'}$, $\frac{c}{c'}$ be in A. P., shew that a', b', c' are in G. P.

194. If the first and last terms of an A.P. be 6 and 61, and the sum 402, find the number of terms.

195. A and B distribute £30 each among the poor. A relieves 20 persons more than B, but B gives to each

person 5*s*. more than A does; how many does each relieve?

196. Two numbers are composed of the same two digits; the difference of the numbers is 9, and of their squares 891; find the numbers.

197. If a, b, c, d be in geometrical progression, shew that $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$.

198. If $x=\frac{b-c}{a}$, $y=\frac{c-a}{b}$, $z=\frac{a-b}{c}$,

prove that $xyz+x+y+z=0$.

199. A farmer bought a certain number of sheep for £30. If he made a profit of 20 per cent. by selling all but five of them for £27, find how many he bought.

200. A testator bequeathed two-fifths of his property to his wife, and one-half of it to his son; one-half of the remainder to his nephew, and the rest to his servants. His wife paid no legacy duty; but the son, the nephew, and the servants paid it at the rates of 1, 3, and 10 per cent. respectively. If the whole legacy duty from the estate amounted to £115, find how much the testator left.

201. Add together $ax^2+by^2+cz^2$; $(b-a)x^2+(a-b)y^2+(b-c)z^2$; $(c-b)x^2+(c-a)y^2+(c-b)z^2$; $c(x^2-y^2-z^2)$.

202. Determine by inspection the quotient when the product of $x^2-2xy+y^2$ and $x^4-2x^2y^2+y^4$ is divided by $x^3-x^2y-xy^2+y^3$.

203. Reduce

$$\frac{2}{x^2-1}+\frac{3}{x-1}+\frac{4}{x+1}-\frac{3}{x^2+1}-\frac{6}{x^4-1}$$

to one fraction.

204. Solve the equations:

(i) $\dfrac{x+2}{x-1}+\dfrac{x+1}{x-2}=\dfrac{x+1}{x+2}+\dfrac{2x+14}{2x-2}$.

(ii) $\dfrac{x-2}{x}+\dfrac{x}{x-2}=\dfrac{5}{2}$. (iii) $x^3+7x=60$.

(iv) $\left.\begin{aligned}(x+1)(y+2)&=(x+6)y,\\ 3x&=5y+2.\end{aligned}\right\}$

205. The product of two numbers is 216, and their sum is 30; find the numbers.

206. From a certain sum I took away a fourth part and put instead £150; I then gave away one-third of this amount, and spent £250 besides; I then found I had £50 left; what had I at first?

207. If α and β are the roots of $ax^2+bx+c=0$, shew that $\alpha^2+\beta^2=\dfrac{b^2-2ac}{a^2}$.

Form the equation whose roots are 3, $-\frac{1}{3}$, and 0.

208. Sum to n terms (1) 1, 4, 7, &c.; (2) $-\frac{3}{2}$, $\frac{1}{4}$, 2, &c.

209. Insert two geometric means between $\frac{2}{3}a^3$ and $18b^3$.

210. In how many ways can a crew of eight rowers be arranged in a boat? In how many ways can they be arranged if only two of them can row stroke?

211. If $a=\dfrac{16}{9}$, $b=1$, $c=\dfrac{5}{3}$, prove that

$$(a+\sqrt{b})(\sqrt{a}-b)(\sqrt{a+b})=\frac{1}{3}\cdot\frac{c^3}{\sqrt{c^3-a}}.$$

212. Divide

$$1-5x+\frac{152}{15}x^2-\frac{106}{225}x^4-\frac{28}{9}x^5 \text{ by } 1-x-\frac{14}{15}x^2.$$

213. Find the G.C.M. of

$$(a+2c)(a-c)-b(4a-3b-c),$$

and

$$(a+2b)(a-b)-c(4a-b-3c).$$

214. Multiply together $x^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{1}{5}}+y^{\frac{2}{5}}$, $x^{\frac{1}{5}}-y^{\frac{1}{5}}$, and $x^{\frac{1}{5}}+y^{\frac{1}{5}}$.

215. Find the L.C.M. of

$$a^{\sqrt{8}}-b^{\sqrt{8}} \text{ and } a^{\sqrt{8}}-3a^{\sqrt{2}}b^{\sqrt{2}}+2b^{\sqrt{8}}.$$

216. Form an equation having for its roots $\frac{5}{2}$ and -3; and another having for its roots $1 \pm \sqrt{2}$.

217. How many pounds of sugar at $4\frac{1}{8}d.$ per lb. must a grocer mix with 110 lbs. at $3\frac{3}{4}d.$ in order that he may gain 20 per cent. by selling the whole at 5*s.* 6*d.* per stone?

218. A crew rowing against stream can do a certain course in $8\frac{1}{2}$ minutes: if there were no stream they could row the distance in 7 minutes less than it takes them to drift down the stream. How quickly would they row down the stream?

219. The sum of a geometric series is $700\frac{5}{9}$, the first term 243, and the common ratio $\frac{2}{3}$; find the number of terms.

220. A father is 54, and his son 27 years of age, how long is it since the ratio of their ages was the duplicate of the present ratio?

221. Simplify $\frac{1}{a-2b}-\frac{4}{a-b}+\frac{6}{a}-\frac{4}{a+b}+\frac{1}{a+2b}$.

222. If $z=\frac{\sqrt{ay^2-a^2}}{y}$ and $y=\frac{\sqrt{ax^2-a^2}}{x}$,

find the value of $\frac{\sqrt{az^2-a^2}}{z}$ in terms of x.

223. Extract the square root of

$$\frac{x^4}{a^4}+\frac{a^4}{x^4}+\frac{x^2}{a^2}+\frac{a^2}{x^2}+2\left(\frac{x^3}{a^3}-\frac{a^3}{x^3}\right)-2\left(\frac{x}{a}-\frac{a}{x}\right).$$

224. Divide $x^{\frac{3}{4}}-2a^{\frac{3}{2}}x^{\frac{3}{8}}+a^3$ by $x^{\frac{1}{4}}-2a^{\frac{1}{2}}x^{\frac{1}{8}}+a$.

225. Solve the equations:

(i) $4\left(\frac{x}{2}+\frac{2}{y}\right)=5$; $3\left(\frac{x}{3}+\frac{3}{y}\right)=5$.

(ii) $\frac{x^2(x-3)+54}{x+3}=10$.

(iii) $9x+2x\sqrt{9x+4}=15x^2-4$.

226. A vessel is filled with a mixture of spirit and water in which there is 81 per cent. of spirit; six gallons are drawn off and the vessel is filled up again with water: the proportion of spirit is now found to be $67\frac{1}{2}$ per cent.; find the content of the vessel.

227. *A* spent 20 guineas in cigars, and found that if he had got one box more for the same money, each box would have been 4*s*. 8*d*. cheaper. How many boxes did he buy?

228. Find the value of $\frac{1}{2}+\frac{3}{4}+1\ldots$ to 90 terms.

229. Find the sum of

(i) the arithmetical series whose first term is m and common difference $\frac{(m-n)^2}{2n(n-1)}$, to n terms.

(ii) the geometrical series whose first term is mn^n and common ratio $\frac{1}{n}$, to n terms.

230. Shew that 441 must represent a square number in any scale of which the base is greater than 4.

231. Multiply $x+\frac{1-\sqrt{3}\sqrt{-1}}{2}y$ by $x+\frac{1+\sqrt{3}\sqrt{-1}}{2}y$.

232. Divide $1+x^3+y^3-3yz$ by $1+y+z$.

233. Simplify $\frac{(x^3-6x^2+11x-6)^{\frac{7}{6}}}{(x-1)^{\frac{2}{3}}(x-2)^{\frac{5}{6}}(x-3)^{1\frac{1}{2}}}$.

234. If $\frac{a^2-b^2-c^2}{b-c}=\frac{a^2-b^2+c^2}{a+c}$,

find the value of either fraction in terms of c.

235. Simplify $\frac{5\sqrt{2}-2\sqrt{3}}{2\sqrt{18}-\sqrt{27}}$, and extract the square root of $2\frac{1}{3}-\frac{2}{3}\sqrt{2}$.

236. Transform 119716 into the scale 7 and extract its square root in that scale.

237. Three numbers, whose sum is 49, are in G.P.; but if each be increased by 15, they will be in H.P. Find them.

238. Solve the equations:

(1) $\frac{b+c}{x-a}+\frac{c+a}{x-b}+\frac{a+b}{x-c}=3$.

(2) $\left(x+\frac{1}{3}\right)^{\frac{1}{2}}-\left(x+\frac{2}{3}\right)^{\frac{1}{2}}=(x+1)^{\frac{1}{2}}$.

(3) $\sqrt{1-x^2}-2x(1+x)\sqrt{1-x}=\frac{19}{9}\sqrt{1-x}$.

239. How many different arrangements can be made of all the letters of the word *appeal*? How many of all the letters of the word *facetious*, the vowels being always in the order of the alphabet?

240. Expand to five terms $(1+a)^{-3}$ and $(1+3a)^{-5}$.

241. Simplify

$$\frac{y^2+z^2-x^2}{yz(x-y)(x-z)}+\frac{z^2+x^2-y^2}{zx(y-z)(y-x)}+\frac{x^2+y^2-z^2}{xy(z-x)(z-y)}.$$

242. Add together $4\sqrt{147}$, $5\sqrt{3}$, and $7\sqrt{27}$; and simplify

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}-\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}.$$

243. Extract the square root of $52+14\sqrt{3}$; and simplify

$$a^{-\frac{pq}{n}}\times b^{\frac{pq}{m}}\left\{\left(\frac{a^{-m}}{b^{-n}}\right)^{\frac{p}{m}}\right\}^{-\frac{q}{n}}.$$

244. If $x=2+3^{\frac{1}{2}}$, find the value of

$$\frac{1-2x}{1-(2x)^{\frac{1}{2}}}+\frac{1+2x}{1+(2x)^{\frac{1}{2}}}.$$

245. If the m^{th} term of an arithmetic progression be a and the n^{th} term be b, find the sums of $(m+n)$ terms and $(m-n)$ terms.

246. Subtract 27541 from 36430 in the scale of 8, and find in what scale 4954 is written 20305.

247. A man borrows every year £25, upon which he pays interest at the rate of 4 per cent. per annum; in how long a time will the interest amount to £91?

248. Prove that

$$(b-c)^6+(c-a)^6+(a-b)^6$$
$$=3(b-c)^2(c-a)^2(a-b)^2+2(a^2+b^2+c^2-bc-ca-ab)^3.$$

249. If P, Q, R be the p^{th}, q^{th}, and r^{th} terms of a harmonic progression, prove that

$$\frac{q-r}{P}+\frac{r-p}{Q}+\frac{p-q}{R}=0.$$

250. Solve the equations:

(1) $\dfrac{x-a}{x-a-c}-\dfrac{x-a+c}{x-a}=\dfrac{x-b}{x-b+c}-\dfrac{x-b-c}{x-b}$.

(2) $ax+by=xy=cx+dy$.

(3) $(x-a)^4+(x-b)^4=(a-b)^4$.

EXAMINATION PAPERS.

I.

1. If $a=2$, $b=3$, $x=6$, $y=5$, find the value of
$$a+2x-\{b+y-[a-x-(b-2y)]\}.$$

2. Find the G. C. M. and the L. C. M. of
$$x^3-x^2-2x+2 \text{ and } x^4-3x^3+2x^2+x-1.$$

3. Solve the equations:

(i) $ax+c=\dfrac{x}{a}+\dfrac{1}{c}$.

(ii) $34x^2-111x+65=0$.

4. Simplify
$$\left\{\frac{a^2}{b^3}+\frac{1}{a}\right\}\div\left\{\frac{a}{b^2}-\frac{1}{b}+\frac{1}{a}\right\}.$$

5. Extract the cube root of 167·284151; and the square root of ·2 to five places of decimals.

6. Divide $x^{\frac{3n}{2}}-y^{-\frac{3n}{2}}$ by $x^{\frac{n}{2}}-y^{-\frac{n}{2}}$.

7. Find the square root of $7-2\sqrt{10}$.

8. Find the arithmetic series whose first term is $\frac{1}{2}$, last term 25, and sum 102.

9. A house of 11 boys contains 6 who can only row, and 3 who can only steer; the other two can do either: how many different crews can they make for their four-oar?

10. Express 625 in the quaternary scale, and extract its square root in that scale.

11. Find the first three terms in the expansion of

$$\left(2-\frac{3x}{4}\right)^5.$$

12. A person having 6 hours 40 min. to spare, wishes to know how far he may take a carriage at $7\frac{1}{2}$ miles per hour, so that walking back at $3\frac{3}{4}$ miles per hour he may reach home one minute before his time expires.

II.

1. If $a=4,\ b=3,\ c=1,\ d=0,$

find the numerical value of

$$3a^2+2bc+4cd+\frac{a^2c-b^2d}{b+c+d}.$$

2. Find the cube of $2a^2+3a-1$.

3. Find the G. C. M. of

$$5x^3+2x^2-15x-6 \text{ and } 7x^3-4x^2-21x+12.$$

4. Resolve into factors

$$a^4-81b^4 \text{ and } 6x^3-17x^2+12x.$$

5. Find the value of

$$\frac{2x}{x+a}+\frac{x^2-ax}{x^2-ax+a^2}-\frac{3x^3}{x^3+a^3}.$$

6. Solve the equations:

(i) $5x+3y=65$; $2y-z=11$; $3x+4z=57$.

(ii) $9x^2+4y^2=40$; $3x-2y=4$.

7. Solve the equations:

(i) $x^2-x+5\sqrt{2x^2-5x+6}=\frac{3}{2}(x+11)$.

(ii) $\dfrac{\sqrt{3x+7}+5}{\sqrt{3x+7}-5}=\dfrac{8+\sqrt{x+6}}{8-\sqrt{x+6}}$.

8. Extract the cube root of

$$8a^6-36a^5+102a^4-171a^3+204a^2-144a+64.$$

9. Divide

$$x^{\frac{5}{2}}-x^3-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}} \text{ by } x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2.$$

10. A farmer mixes barley at 2*s.* 4*d.* a bushel with rye at 3*s.* a bushel and wheat at 4*s.* a bushel, so that the whole is 100 bushels, and worth 3*s.* 4*d.* a bushel. Had he put twice as much rye and 10 bushels more of wheat, the whole would have been worth exactly the same per bushel: how much of each kind was there?

11. Simplify

$$\left\{1-\frac{y^2}{(x+y)^2}\right\}\left\{1-\frac{2xy}{x^2+4y^2}\right\}\left\{x^3+8y^3+\frac{(x^3-8y^3)^2}{x^3+8y^3}\right\}.$$

12. Find the difference between the sum of the squares of any three quantities, and the square of their sum.

13. Insert five geometric means between $\frac{8}{9}$ and $10\frac{1}{8}$; and three harmonical means between $\frac{1}{3}$ and $\frac{1}{5}$.

14. If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; again, if $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, shew that $\frac{a}{b}=\frac{c}{d}$.

15. Find the number of possible arrangements of all the letters of the word *measles*.

III.

1. DEFINE the terms "coefficient," "power," "index;" and prove that $a^3 \times a^4 = a^7$.

2. If $s=\frac{a+b+c}{2}$, find the value of

$$abc+2(s-a)(s-b)(s-c)-a(s-a)^2-b(s-b)^2-c(s-c)^2,$$

when $a=1,\ b=2,\ c=3$.

3. Multiply

$$a^2+b^2+c^2-bc-ca-ab \text{ by } a+b+c,$$

and

$$(b+c)^2-a^2 \text{ by } a^2-(b-c)^2.$$

4. Write down the factors of

$$x^2-y^2,\ x^3+y^3,\ x^4+2x^2y^2+y^4, \text{ and } x^2+(a+b)x+ab.$$

5. Find the G.C.M. of

$$x^5-2x^4+3x^3-3x^2+2x-1 \text{ and } x^4+x^3+2x^2+x+1.$$

6. Simplify

(i) $\frac{2ab}{a^3-b^3}+\frac{a+b}{a^2-b^2}+\frac{1}{a-b}$.

(ii) $\left(\frac{mx+ny}{mx-ny}+\frac{mx-ny}{mx+ny}\right) \div \left(\frac{mx+ny}{mx-ny}-\frac{mx-ny}{mx+ny}\right)$.

7. Extract the square root of

$$\frac{a}{x^{\frac{2}{3}}}-\frac{2a^{\frac{3}{4}}b^{\frac{1}{4}}}{x^{\frac{1}{2}}y^{\frac{1}{6}}}+\frac{3a^{\frac{1}{2}}b^{\frac{1}{2}}}{x^{\frac{1}{3}}y^{\frac{1}{3}}}-\frac{2a^{\frac{1}{4}}b^{\frac{3}{4}}}{x^{\frac{1}{6}}y^{\frac{1}{2}}}+\frac{b}{y^{\frac{2}{3}}}.$$

8. Solve the following equations:

(1) $$\frac{2x+7}{3(x-3)}-\frac{x+6}{2x-3}=3.$$

(2) $$(x+2)^{\frac{1}{2}}+(2x+11)^{\frac{1}{2}}=(3x+43)^{\frac{1}{2}}.$$

(3) $$\left.\begin{aligned} x^2-xy&=6,\\ 2y^2-xy&=20. \end{aligned}\right\}$$

If $a : b :: c : d$, prove that

$$ma^2+nb^2 : pa^2+qb^2 :: mc^2+nd^2 : pc^2+qd^2.$$

10. The interest of a certain sum of money, for a certain number of years, is £75, which is also the interest of a sum less than the former by £200, for two more years; find the sums, allowing simple interest at 5 per cent. per annum.

11. A number consists of three digits, the first two being the same, and is equal to the product of the sum of the digits, and of the number formed by the last two digits; if the product of the last two digits be 10, find the number.

12. If $x \propto y^{\frac{3}{2}}$ and $\sqrt{xy} \propto yz$, shew that $x \propto z^6$.

IV.

Cambridge Previous Examination. June, 1874.

1. STATE the rule for the addition of algebraical quantities.

Add together

$$2(a-2b+3c),\quad 3(2a+3b-4c),\quad 4(-a+b-2c).$$

2. Prove that $a-(b-c)=a-b+c$, and simplify the expressions

$$4\left(x-\frac{y}{2}\right)-6\left[\left(2x-\frac{y}{3}\right)-12\left(\frac{x}{3}-\frac{3x+2y}{24}\right)\right].$$

3. Multiply

$$\frac{x^4}{16}-\frac{x^3}{2}+\frac{3x^2}{2}-2x+1 \text{ by } \frac{x^2}{4}-x+1;$$

and divide the product by

$$\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1.$$

4. Divide

$$ab\,(x^2+y^2)+(a^2+b^2)\,xy+(a-b)\,(x-y)-1 \text{ by } ax+by-1.$$

5. Shew that the expression $\frac{1}{x-a}+\frac{1}{x-b}$ has the same value when $x=a+b$, and when $x=\frac{2ab}{a+b}$.

6. Define a^m, when m is a positive integer.

Prove that $\frac{a^7}{a^3}=a^4$, and that $(a^5)^3=(a^3)^5$.

7. Simplify the fractions:

(1) $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

(2) $\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}\times\frac{\frac{1}{y}-\frac{1}{x}}{\frac{x^2}{y}-\frac{y^2}{x}}$.

(3) $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}$.

8. State the rule for finding the greatest common measure of two algebraical quantities, and find that of x^7+1 and $x^{12}+x^3$.

Reduce to its lowest terms the fraction

$$\frac{x^3-8x^2y+17xy^2-6y^3}{2x^3-9x^2y+10xy^2-3y^3}.$$

9. Prove that

$$(a^2-ax+x^2)(a^2+ax+x^2)=a^4+a^2x^2+x^4,$$

and resolve $a^{16}+a^8x^8+x^{16}$ into four factors.

10. Solve the following equations:

(1) $\frac{3}{4}(2x-7)+\frac{x+2}{4}=\frac{3}{8}(3x+10)-6.$

(2) $2x-1-2(3x-2)+3(4x-3)-4(5x-4)+5(6x-5)=0.$

(3) $(x+1)(x+2)(x+3)=(x+4)(x+5)(x-3).$

(4) $\begin{cases} \frac{3x}{4}-\frac{5y}{7}=-1, \\ \frac{2x}{3}-\frac{y}{2}=1. \end{cases}$

11. Define ratio, and prove that if $a : b :: c : d$,

$$ma+nb : ma-nb :: mc+nd : mc-nd.$$

Prove that a ratio of greater inequality is diminished by adding any positive quantity to both terms of the ratio.

12. Solve the equations:

(1) $(2x-3)(3x+4)=39.$

(2) $\begin{cases} \frac{x}{y}-\frac{y}{x}=\frac{3}{2}, \\ x+y=6. \end{cases}$

(3) $\begin{cases} 2x^2+3xy=27, \\ xy+y^2=4. \end{cases}$

V.

Cambridge Previous Examination. December 7, 1875.

ELEMENTARY ALGEBRA.

1. WHAT are the numbers eight and two called, respectively, in the expression $8a^2$, and how are they treated in division?

2. Simplify $b^2+[a(a-b)-\{ab-b(a+b)\}]$,

and find the value of $x^2-3x^{-1}+\sqrt{x}$

when $x=1$.

3. Shew that

$$a(a+b-c)(a-b+c)-b(-a+b+c)(a+b-c)$$
$$=(a-b)\{(a+b)^2-c^2\};$$

and resolve $x^4-3x^2y^2+y^4$ into factors.

4. Divide x^4-x^2+2x-1 by x^2+x-1,

and find the remainder in the division of x^3-y^3 by $x+y$.

5. Find the greatest common measure of

$$a^3+a+2 \text{ and } a^3+2a+3.$$

6. Reduce to their simplest terms the fractions

$$\frac{y}{2}\left(\frac{1}{x+y}+\frac{1}{x-y}\right)\cdot\frac{x^2-y^2}{x^2y+xy^2}, \qquad \frac{1}{a-\dfrac{a^2+1}{a-\dfrac{1}{a-1}}}.$$

7. Solve the equations:

(i) $\frac{x}{2}+\frac{x}{3}=10$; (ii) $\frac{x+5}{x-1}+\frac{x+1}{x-5}=2$;

(iii) $\left.\begin{aligned}2x-3y&=42,\\ x+4y&=41.\end{aligned}\right\}$

8. Solve the equations:

(i) $x^2-4x=5$; (ii) $\sqrt{2x-5}+\sqrt{x+1}=9$;

(iii) $\left.\begin{aligned} x^2-xy&=24,\\ y^2+xy&=65.\end{aligned}\right\}$

9. Shew that a ratio of less inequality is increased by adding the same quantity to each term.

The terms of a ratio are three and seven, what number must be added to each in order that the ratio may be doubled?

10. If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$, prove that $\frac{c_1a_1^2+c_2a_2^2+c_3a_3^2}{c_1b_1^2+c_2b_2^2+c_3b_3^2}=\frac{a_1^2}{b_1^2}$.

11. When is one quantity said to vary as another?

If z varies as x^3 when y is constant, and as y when x is constant, how will it vary when neither x nor y is constant?

VI.

Cambridge Previous Examination. December 8, 1875.

Higher Algebra.

1. When are quantities said to be in arithmetical progression? Write down the first five terms of an arithmetical series whose first term is 6, and whose tenth term is 20.

2. Prove that the sum of n terms of an arithmetical series is $\frac{n}{2}\{2a+(n-1)d\}$, a being the first term and d the common difference.

The sum of a certain number of terms of an arithmetical series is 36, and the first and last of these terms are 1 and 11 respectively: find the number of terms and the common difference of the series.

3. What is meant by the *geometric* mean between two quantities? Is it *greater* or *less* than the *arithmetic* mean between the same quantities?

4. Sum to 12 terms the series

$$\frac{5}{4}+\frac{5}{2}+5+10+\ldots\ldots$$

5. Prove that the sum of two quantities divided by the sum of their reciprocals is equal to the product of the quantities.

6. A farmer bought a certain number of sheep for £30. If he made a profit of 20 per cent. by selling 15 of them for £27, find how many he bought.

7. A testator bequeathed one-fifth of his property to each of his four children, and the remainder in equal parts to a nephew and a cousin. The children, the nephew, and the cousin, paid legacy duty at the rates of 1, 3, and 5 per cent. respectively, and the whole legacy duty from the estate amounted to £160. Find how much the testator left.

8. If $x=\frac{b+c}{a}$, $y=\frac{c+a}{b}$, $z=\frac{a+b}{c}$,

prove that $x+y+z-xyz+2=0$.

VII.

Cambridge; Second General Examination for the Ordinary B.A. Degree.

November 29, 1875. 1—4 P.M.

1. SIMPLIFY

$$\frac{1}{2}\frac{a^2+x^2}{a^2-x^2}-\frac{1}{2}\frac{a+x}{a-x}-\left(\frac{a}{a+x}\right)^2,$$

and prove that

$$\frac{x^2+1}{x^2-1}-1\div\left(\frac{2x}{2x-1}-\frac{2}{2-x}\right)=\frac{5x}{2(x^2-1)}.$$

2. Solve the equations:

(1) $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x-1}{15}$.

(2) $\left.\begin{aligned}\frac{x}{3}&=5-2y,\\ \frac{2x-1}{5}-y+1&=0\end{aligned}\right\}$.

3. Solve the equations:

(1) $\frac{3}{x-7}-\frac{4}{x-3}=\frac{5}{6}$.

(2) $(x-7)(x+3)-\sqrt{(x+4)(x-8)}=67$.

(3) $\left.\begin{aligned}x^2&=ax+by\\ y^2&=ay+bx\end{aligned}\right\}$.

4. Find the sum of an arithmetical series whose first term, difference, and number of terms are given.

The first and last terms of an arithmetical series are 5 and 68, the sum is 365, find the number of terms.

5. Find the sum of a series of n terms in geometrical progression. Shew that a recurring decimal may be regarded as a geometrical progression. Prove that

$$\sqrt{1\cdot36\dot{1}}=1\cdot1\dot{6}.$$

6. Explain what is meant by the term variation.

The value of diamonds varies as the squares of their weight; two diamond rings are worth respectively £50 and £25, and the weights of the diamonds are three and two carats; the value of the gold is the same in each; what is that value?

7. If α, β be the roots of the equation

$$x^2-px+q=0,$$

shew that $x^2-px+q=(x-\alpha)(x-\beta)$.

Resolve the expression

$$39x^3 - 121x - 22$$

into its two factors.

8. Define proportion; if $a : b :: c : d$, shew that

$$\frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

If a be the greatest of the four magnitudes a, b, c, d, shew that $a-b > c-d$.

9. A and B distribute ten guineas each among the poor, A relieves six more people than B, but B gives to each person four shillings more than A does; how many does each relieve?

10. It is between three and four o'clock, and in a quarter of an hour the minute-hand will be as much in front of the hour-hand as it is now behind it; what is the time?

11. Two numbers are composed of the same two digits; the difference of the numbers is 18, and of their squares 1584; find the numbers.

12. If four numbers be in proportion, and the first three be in arithmetic progression, shew that the reciprocals of the last three are in arithmetic progression also.

VIII.

Oxford; Responsions.

1. WHAT is the value of

$$a^0,\ a^1,\ \frac{a}{o},\ \frac{o}{a}, \text{ and of}$$

$$\sqrt[3]{x}\left\{y^3 + \sqrt[4]{\frac{x^2+2xy+y^2}{y^2}}\right\} \text{ when } \begin{cases} x=8, \\ y=1 \end{cases}?$$

2. Multiply

$$x^3+3ax(x+a)+a^3 \text{ by } x-a,$$

and divide

$$(a^2-2ab+b^2)(x^3+a^3) \text{ by } ax-bx+a^2-ab.$$

3. Find the G. C. M. of

$$2x^2+3xy+6x+9y \text{ and } 3x^2-2xy+9x-6y.$$

4. Find the L. C. M. of

$$x^2-1,\ x^2-4y^2, \text{ and } x^2+x-2xy-2y.$$

5. Simplify

$$\cfrac{x}{1+\cfrac{x}{1+x+\cfrac{x}{1+x+x^2}}}$$

and

$$\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2y+y^3}{x^3-y^3}.$$

6. Find the square root of

$$\left(x^2+\frac{x^2}{y}-2x\right)\left(1+\frac{1}{y}\right)+1.$$

7. Find two numbers in the proportion of 2 to 3, the sum of whose squares is 117.

8. Solve the following equations:

(1) $\frac{1}{12}x-\frac{1}{8}(8-x)-\frac{1}{4}(5+x)+\frac{11}{4}=0;$

(2) $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2;$

(3) $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1;$

(4) $\left.\begin{aligned} x+ay&=b,\\ ax-by&=c. \end{aligned}\right\}$

9. *A* had twice as much money as *B*; but when *B* had won £5 from *A*, he had 3 times as much as *A*. How much had they each at first?

10. Find a fraction, such that if 5 be added to the numerator, it $=1$; but if 3 be taken from the denominator, it $=\frac{1}{2}$.

IX.

Oxford; Second Public Examination.

(I.)

1. MULTIPLY together

(1) $(x^2+xy+y^2)(x^2-xy+y^2)$;

(2) $(x+y+z-\sqrt{yz}-\sqrt{zx}-\sqrt{xy})(\sqrt{x}+\sqrt{y}+\sqrt{z})$.

2. Divide

(1) $9x^4-52x^2y^2+64y^4$ by $3x-4y$;

(2) $9a^{4n}-\frac{1}{4}$ by $2a^n+1$.

3. Simplify

(1) $\left(\frac{1}{x}+\frac{x-1}{2-x}\right)\div\left(\frac{1}{2-x}-\frac{x-1}{x}\right)$;

(2) $\frac{4ab(a^4-b^4)}{a^6-b^6}-\frac{a^2-ab+b^2}{a^2+ab+b^2}+\frac{a^2+ab+b^2}{a^2-ab+b^2}$;

(3) $\frac{a+\frac{ab}{c+d}}{c+\frac{cd}{a+b}}\times\frac{a+b+d}{b+c+d}\div\frac{a+b}{c+d}$.

4. Solve the following equations:

(1) $\frac{3x+9}{5}+\frac{9x+3}{37}=x$;

(2) $3(x+a)(x-b)+2(x-a)(x-b)=5x^2$;

(3) $x\sqrt{a-x}+\sqrt{a}\sqrt{ax-x^2}=(a-x)\sqrt{x}$;

(4) $\begin{cases}4x-3y=6,\\5x+6y=27.\end{cases}$

5. Prove the rule for finding the greatest common measure of two quantities, and find the G.C.M. of

$$8(x+1)(x^2-4) \text{ and } 4(x+1)(x^2-5x-14).$$

6. Find the square root of

(1) $16a^{10}b^8c^6d^{-4}e^{-2}$;

(2) $x^2+8x-64x^{-1}+64x^{-2}$;

(3) $x^2y^2+2y^2+2xy+2\frac{y}{x}+\frac{y^2}{x^2}+1$.

7. Shew that it is possible to find the square root of $a+\sqrt{b}$ if a^2-b be a square.

8. A market woman sold half her eggs at the rate of four for sixpence, two added to half the remainder for a penny a-piece, and the rest at the rate of eight for sixpence. Her receipts were four shillings. How many eggs were there at first?

9. Prove that

$$\sqrt{-\sqrt{3}+\sqrt{3+8\sqrt{7+4\sqrt{3}}}}=2.$$

10. Find the least common multiple of

$$(1+x)(1+x^2)(1+x^3), \quad (1-x)(1-x^2)(1-x^3),$$
$$1+x^3, \text{ and } 1-x^6.$$

X.

Oxford; Second Public Examination.

(II.)

1. Prove that the product of the two roots of a quadratic equation is equal to the last term.

The product of two roots of an equation is -18, and the difference between them is 9; form the equation.

2. Solve the equations:

(1) $12x^2-8x-15=0$;

(2) $\begin{cases}(x+2y)^2+(2x-y)^2=85;\\ xy\qquad\qquad\quad =4;\end{cases}$

(3) $5\sqrt{\frac{3}{x}}+7\sqrt{\frac{x}{3}}=22\frac{2}{3}$.

3. Sum to n terms, and if possible to infinity:

$1,\quad 1\frac{1}{2},\quad 2,\ \ldots\ldots;$

$1,\quad 1\frac{1}{2},\quad 2\frac{1}{4},\ \ldots\ldots;$

$2,\quad 1\frac{1}{2},\quad 1\frac{1}{3},\ \ldots\ldots.$

4. Insert eleven harmonic means between $\frac{1}{4}$ and $\frac{1}{18}$, and find the sum of their reciprocals.

5. If $a:b::c:d$, prove that

$$a+\frac{1}{b}:b+\frac{1}{a}::c+\frac{1}{d}:d+\frac{1}{c}.$$

6. Prove the binomial theorem for positive integral indices, and find the ninth term of

(1) $(a-x)^{14}$, (2) $(3a-7x)^{-\frac{1}{2}}$,

and the middle term of

$$\left(5a-\frac{x}{5}\right)^{16}.$$

7. Prove that the number of permutations of n things taken r together is

$$n(n-1)(n-2)\ldots\ldots(n-r+1).$$

How many permutations can be formed with the letters of the word *propeller*?

8. If α, β are simultaneous values of x, y, which satisfy $ax+by-c=0$, find expressions which give all the possible values of x and y.

How can a bill of £5. 4*s*. be paid in half-crowns and half-guineas?

9. Multiply together the duodenary numbers *tete* and *ete*.

Place three digits to the right of the denary number 31237 so as to form a number divisible by 99.

10. How many words of three letters, one of which is a vowel, can be formed out of six consonants and three vowels?

ANSWERS.

Ex. I.

1. 2. 2. 12. 3. 30. 4. 18. 5. 30.
6. 60. 7. 4. 8. 2. 9. 18. 10. 6. 11. 4.
12. 14. 13. 5. 14. 6. 15. 7. 16. 2. 17. 7.
18. 2. 19. 2. 20. 5. 21. 4. 22. 6.
23. -75. 24. -12. 25. 6. 26. -2. 27. 2.
28. 3. 29. 1. 30. 2. 31. 21. 32. 36.
33. 5. 34. 63. 35. 14. 36. 26. 37. 16.
38. 5. 39. -59. 40. 19. 41. 8. 42. -41.
43. 156. 44. 97. 45. 89. 46. 25. 47. 8.
48. 27. 49. -92. 50. -15. 51. -6. 52. $-97\frac{3}{5}$.
53. $125\frac{4}{25}$. 54. $41\frac{2}{3}$. 55. 2. 56. $1\frac{2}{3}$. 57. 5.
58. -2. 59. $\frac{7}{72}$. 60. 74.

Ex. II.

1. $x^{\frac{5}{6}}$. 2. $4x^{\frac{7}{10}}$. 3. $14a^{\frac{5}{6}}b^{\frac{7}{10}}$. 4. $48a^{\frac{5}{6}}b^{\frac{1}{2}}$.
5. $75a^{\frac{5}{4}}b^{\frac{1}{2}}c^{\frac{2}{3}}$. 6. $18x^{\frac{7}{4}}y$. 7. $32a^{2}b^{\frac{7}{5}}c^{2}$. 8. $40x^{\frac{7}{3}}y^{2}z^{\frac{7}{2}}$.
9. $5a^{2}b^{\frac{19}{7}}c^{\frac{12}{15}}d^{\frac{6}{5}}$. 10. $21a^{\frac{13}{3}}b^{\frac{6}{15}}c^{\frac{2}{5}}d\theta^{\frac{13}{5}}$. 11. $2a^{-6}$.
12. $6a^{-9}b^{-5}$. 13. $4a^{-4}b^{-1}$. 14. $15x^{-2}yz^{-8}$.

15. xy^{-6}. 16. $a^{-1}b^{-2}$. 17. $4a^5b^{-\frac{1}{3}9}c^{-1}$. 18. $6a^{-1}c^{-2}$.

19. $6xy^{-1}$. 20. $21a^{-2}b^{-1}d^{-2}$. 21. $12a^{\frac{4}{3}}+8a$.

22. $9a^2+3a^{\frac{4}{3}}+7a^{\frac{2}{3}}$. 23. $18x^{\frac{7}{2}}-6x^{\frac{3}{2}}-2x^{-\frac{1}{2}}$.

24. $-77a^{-\frac{1}{2}}x^{\frac{1}{6}}-44+11a^{\frac{1}{2}}x^{-\frac{1}{6}}$. 25. $x^3-4x^{\frac{4}{5}}y^3$.

26. $3x^{\frac{5}{2}}+x^{\frac{4}{3}}-3x^{\frac{1}{6}}-x^{-1}$. 27. $18a^2-2a^{\frac{2}{3}}b^{\frac{1}{3}}+40a^{\frac{7}{5}}b^{\frac{1}{2}}$.

28. $24a^{\frac{7}{4}}+8a^{\frac{3}{2}}b^{\frac{1}{3}}+a^{-\frac{5}{2}}b$. 29. $48a^{\frac{5}{2}}c^3+33a^{\frac{3}{2}}b^2c^{-3}+45ab^3c^{-6}$.

30. $4x^2y^{-3}-64y$.

31. $12a^{-5}+25a^{-3}b^{-1}+4a^{-1}b^{-2}-13ab^{-3}-4a^3b^{-4}$.

32. $x^{-1}-2-19x+6x^3$. 33. x^3-x^{-2}.

34. $a^{\frac{3}{4}}-2a^{\frac{5}{8}}+4a^{\frac{1}{2}}-64+128a^{-\frac{1}{3}}-256a^{-\frac{1}{4}}$.

35. $x^{\frac{4}{3}}-8+16x^{-\frac{4}{3}}$.

36. $6a^{-4}-17a^{-3}b^{\frac{1}{2}}+17a^{-2}b-6a^{-1}b^{\frac{3}{2}}-6b^2$.

37. $a-2a^{\frac{5}{6}}b^{\frac{1}{3}}-16a^{\frac{1}{3}}b^{\frac{4}{3}}+32a^{\frac{1}{6}}b$.

38. $2x^{\frac{7}{4}}-7x+11x^{\frac{1}{4}}-7x^{-\frac{1}{2}}-3x^{-\frac{5}{4}}$.

39. $12y^{\frac{3}{4}}+5a^{-\frac{1}{2}}y^{\frac{1}{2}}-3a^{-\frac{3}{2}}-4a^{-\frac{5}{2}}y^{-\frac{1}{2}}$.

40. $27a^{\frac{3}{7}}+18a^{\frac{1}{2}}b^{-1}c^{-\frac{1}{5}}-b^{-3}+8c^{-\frac{3}{5}}$. 41. $x^{\frac{1}{2}}$.

42. $3a^{\frac{1}{3}}b^{\frac{1}{2}}$. 43. $3a^{\frac{7}{10}}b^{1\frac{1}{2}}c^{\frac{1}{2}}$. 44. $xy^{\frac{1}{4}}z$.

45. $3x^2y^{\frac{1}{4}}z^5$. 46. $a^{-1}b$. 47. $a^2b^4c^{-2}$.

48. $x^{-4}z^{-8}$. 49. $6a^7bc^{-5}d^2$. 50. $a^{-14}c^8d^{-2}$.

51. $3a^4b^{-3}-5a^3b^{-1}c^{-3}$. 52. $8x^4y+5z^5-3x^3y^{-6}z^{-1}$.

53. $-5x^3+3y^{-4}z^6-x^{-2}y^{-8}z^{12}$.

54. $-3a^4c^{-4}d^{-5}-6b^{-2}c^{-1}d^{-3}+2ab^{-3}c^{-3}$.

55. $5x^{10}yz^{-4}-x^3y^4-3yz^{-6}$.

56. $7x^3y^3z^3+4x^{\frac{3}{2}}z^{1\frac{1}{6}}-3y^{-3}z^{\frac{7}{3}}$. 57. $x^{\frac{3}{2}}+2x$.

58. $4x^{\frac{4}{5}}+y^{\frac{1}{2}}$. 59. $4a^{\frac{2}{5}}-a^{1\frac{1}{10}}$. 60. $3a^{-1}+2$.

61. $3a+5a^{\frac{1}{2}}$. 62. $2x^{\frac{4}{5}}-10x^{\frac{3}{5}}-6x^{\frac{2}{5}}$.

63. $2a^3-4a^2+2a$.
64. $1-4a^{\frac{1}{2}}+4a$.
65. $2a^{\frac{4}{5}}x^{\frac{7}{5}}-10a^{\frac{3}{5}}x^{\frac{3}{5}}+12a^{\frac{2}{5}}x^{\frac{1}{5}}$.
66. $x^2y^{-2}-1+x^{-2}y^2$.
67. $x^{\frac{2}{3}}+y$.
68. $a-a^{\frac{1}{2}}$.
69. $12a^{\frac{4}{3}}-11a+10a^{\frac{2}{3}}-9a^{\frac{1}{3}}+8$.
70. $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.
71. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$.
72. $3-5a^{-1}-5a^{-2}$.
73. $a^{\frac{2}{3}}b^{-\frac{3}{2}}+7a^{\frac{1}{3}}b^{-1}-4b^{-\frac{1}{2}}$.
74. $x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}x^{-\frac{5}{2}}+\frac{1}{4}x^{-\frac{7}{2}}\ldots$
75. $x^{2p}+x^py^q+y^{2q}$.
76. x^n+1+x^{-n}.
77. $4x^{\frac{3}{4}}-3x^{\frac{1}{4}}$.
78. $7a^{-1}-4a^{-\frac{1}{2}}b^{\frac{1}{2}}$.
79. $x^{\frac{1}{3}}-x^{\frac{1}{6}}+1$.
80. $5x^{\frac{2}{3}}-3a^{\frac{1}{4}}x^{\frac{1}{3}}+4a^{\frac{1}{2}}$.
81. $2x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{-1}-2y^{-2}$.
82. $a^2y^{-\frac{2}{3}}-2a^{\frac{1}{2}}y^{\frac{1}{3}}+y^{\frac{4}{3}}$.
83. $2a^{\frac{1}{2}}-5a^{-\frac{1}{2}}x^{\frac{1}{2}}$.
84. $3x^{1\frac{1}{6}}y^{\frac{1}{4}}-x^{-1\frac{3}{6}}y^{-\frac{1}{4}}$.

Ex. III.

1. x^2-y^2.
2. x^2-4y^2.
3. $4x^4-a^2$.
4. $9x^4-16y^4$.
5. $25x^2y^2-z^4$.
6. $9a^4-b^2c^4$.
7. x^4-y^4.
8. a^4-16b^4.
9. a^8-c^8.
10. $81a^4-16$.
11. $2x^8-162a^4$.
12. $48x^4-1875y^8$.
13. $x^2+2xy+y^2$.
14. $k^2+2kl+l^2$.
15. $9a^2+6ab+b^2$.
16. $16a^2-24ab+9b^2$.
17. $36a^2+84ab+49b^2$.
18. $81x^4-36x^2y+4y^2$.
19. $144a^2b^4+120ab^3c+25c^2$.
20. $16x^4y^2z^2+56a^2x^2yz+49a^4$.
21. $9x^2-42xy^2+49y^4$.
22. $16x^4y^2-56x^2yz^3+49z^6$.
23. $20k^4-60k^2l+45l^2$.
24. $12a^4b^2-72a^2bc^3+108c^6$.

25. $81x^4-18x^2y^2+y^4$.
26. $16a^4-392a^2x^2+2401x^4$.
27. $3x^8-6x^4y^4+3y^8$.
28. $512a^8-64a^4c^4+2c^8$.
29. x^4-18x^4.
30. $197a^2-4ax$.
31. x^3+y^3.
32. k^3-l^3.
33. x^6+y^6.
34. x^6-z^6.
35. x^3+8.
36. $8x^3+27$.
37. $64k^3+125l^3$.
38. $8a^6-b^3c^3$.
39. 0.
40. $83x^6-20x^2y^3$.
41. x^2+5x+6.
42. $x^2+12xy+35y^2$.
43. $a^2+15ax+50x^2$.
44. $12a^2+29ab+14b^2$.
45. $x^2-5xy+6y^2$.
46. $x^2-6x-16$.
47. $12x^2-19x-18$.
48. $10x^4-31x^2y+15y^2$.
49. $3a^2-13ax-10x^2$.
50. $14a^2-11ax-15x^2$.
51. $11a^2+32ax-48x^2$.
52. $22a^2-67ax-35x^2$.
53. $12x^4+14x^2yz-6y^2z^2$.
54. $15x^4-80x^2y+105y^2$.
55. $21a^2b^2-14abc^2-56c^4$.
56. $84a^2x^2+4axy^2-8y^4$.
57. $2ax+2by$.
58. $a^2y-4abx-4b^2x+4b^2y$.
59. $4x^2+2x$.
60. $3a^2-6ax+4x^2$.

Ex. IV.

1. $(c+d)(c-d)$.
2. $(a+5)(a-5)$.
3. $(2x+7)(2x-7)$.
4. $(10a+bc^2)(10a-bc^2)$.
5. $(12x^2+yz^3)(12x^2-yz^3)$.
6. $(11a+8b^2c)(11a-8b^2c)$.
7. $x(4a+x)(4a-x)$.
8. $2(4x+y)(4x-y)$.
9. $3y(2x+5y^2)(2x-5y^2)$.
10. $7b(2ab^3+3c^2)(2ab^3-3c^2)$.
11. $(a^2+b^2)(a+b)(a-b)$.
12. $(x^2+y^4)(x+y^2)(x-y^2)$.
13. $(x^2+9y^2)(x+3y)(x-3y)$.
14. $(4a^4+x^2y^6)(2a^2+xy^3)(2a^2-xy^3)$.

15. $(9y^3+25x^4)(3y+5x^2)(3y-5x^2)$.
16. $a^2(x^2+a^2)(x+a)(x-a)$.
17. $x(x^2+y^2)(x+y)(x-y)$.
18. $6x^3(x^2+4y^4)(x+2y^2)(x-2y^2)$.
19. $5a^2(9x^4+a^2y^6)(3x^2+ay^3)(3x^2-ay^3)$.
20. $3a^2b^4(4x^2+9a^2b^2)(2x+3ab)(2x-3ab)$.
21. $(x+y)^2$. 22. $(a^2+b^3)^2$. 23. $x^2(x^2+y^3)^2$.
24. $(a-b)^2$. 25. $(a^3-b^4)^2$. 26. $a^2(1-3ab)^2$.
27. $x^2(6-5xy^3)^2$. 28. $a^2x^6(12+7a^2x)^2$.
29. $y^2(11x^5-6y^2z^3)^2$. 30. $a^2(9x^3-2ay^2z)^2$.
31. $3(5x^2-4y^3)^2$. 32. $7ax^4(10-a^2x^2)^2$.
33. $(a+x)(a^2-ax+x^2)$. 34. $(a^2+x^2)(a^4-a^2x^2+x^4)$.
35. $(a-2b)(a^2+2ab+4b^2)$.
36. $(3x+4)(9x^2-12x+16)$.
37. $(2x-y)(4x^2+2xy+y^2)$.
38. $(5x^2+3yz^3)(25x^4-15x^2yz^3+9y^2z^6)$.
39. $8x^2(2x^3-y^2)(4x^6+2x^3y^2+y^4)$.
40. $3x^2y^3(2-5xyz^3)(4+10xyz^3+25x^2y^2z^6)$.
41. $(x+6)(x+1)$. 42. $(x+7)(x+6)$.
43. $(x+10)(x+2)$. 44. $(x+15)(x+3)$.
45. $(x-9)(x-3)$. 46. $(x-10)(x-3)$.
47. $(x-36)(x-3)$. 48. $(x-53)(x-4)$.
49. $(x-4)(x+3)$. 50. $(x-10)(x+7)$.
51. $(x-16)(x+1)$. 52. $(x+7)(x-5)$.
53. $(x+12)(x-5)$. 54. $(x+13)(x-1)$.
55. $(2x+1)(x+3)$. 56. $(3x+4)(x+2)$.
57. $(2x+5)(2x+3)$. 58. $(3x+5)(4x+5)$.

59. $(5x^3-4)(3x^3-1)$.
60. $(4a^2-3x)(5a^2-7x)$.
61. $(3x-11y)(x+y)$.
62. $3b^2(3a^2-b)(a^2+4b)$.
63. $35a(a+2x^2)(a+x^2)$.
64. $5x^2(21x+20y^3)(x-y^3)$.
65. $(a+b)(x+y)$.
66. $(a+b)(x-y)$.
67. $(a-b)(x^2-y^2)$.
68. $(cd-ef)(3b+1)$.
69. $2(2l+k)(ck-m)$.
70. $(2a^2x^2-3b^2y^2)(x+y)$.
71. $(abx+cdy-efz)(ax+by)$.
72. $(a-b)(a+b+1)$.
73. $3(x+y)(x-y+2)$.
74. $(x^2+2y^2)(x^2-2y^2+1)$.
75. $2x(x-y)$.
76. $4ab$.
77. $-y(4x+3y)$.
78. $24xy(x^2+9y^2)$.
79. $2b(3a^2+b^2)$.
80. $(a+b+c)(a+b-c)(a-b+c)(b+c-a)$.
81. $16ab(a^2+3b^2)$.
82. $x^3(5x^2+4y^2)(x-8y)$.
83. $(a+b-6c)(a+b-5c)$.
84. $2b(a^2-b^2)$.

Ex. V.

1. $x^3+3x^2y+3xy^2+y^3$.
2. $a^6+3a^4b+3a^2b^2+b^3$.
3. $k^3-3k^2l+3kl^2-l^3$.
4. $8x^3-12x^2y+6xy^2-y^3$.
5. $64x^6+48x^4yz+12x^2y^2z^2+y^3z^3$.
6. $125a^3-300a^2b^2+240ab^4-64b^6$.
7. $216x^6+540x^4yz^3+450x^2y^2z^6+125y^3z^9$.
8. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
9. $c^4-4c^3d+6c^2d^2-4cd^3+d^4$.
10. $a^4+8a^3d+24a^2d^2+32ad^3+16d^4$.
11. $625a^4-500a^3c^2+150a^2c^4-20ac^6+c^8$.
12. $256x^8+256x^6yz^2+96x^4y^2z^4+16x^2y^3z^6+y^4z^8$.
13. $16x^{12}-160x^9yz^2+600x^6y^2z^4-1000x^3y^3z^6+625y^4z^8$.

14. $81x^4-432x^3y+864x^2y^2-768xy^3+256y^4$.

15. $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$.

16. $a^5-5a^4c+10a^3c^2-10a^2c^3+5ac^4-c^5$.

17. $32x^{10}-80x^8y^2+80x^6y^4-40x^4y^6+10x^2y^8-y^{10}$.

18. $k^5+15k^4l+90k^3l^2+270k^2l^3+405kl^4+243l^5$.

19. $32a^5-240a^4b^2+720a^3b^4-1080a^2b^6+810ab^8-243b^{10}$.

20. $a^{15}-10a^{12}bc^2+40x^9b^2c^4-80a^6b^3c^6+80a^3b^4c^8-32b^5c^{10}$.

21. $x^2+2xy+2xz+y^2+2yz+z^2$.

22. $x^4+2x^2y^2+2x^2z^2+y^4+2y^2z^2+z^4$.

23. $x^4+2x^3y+3x^2y^2+2xy^3+y^4$.

24. $a^4-2a^2b^2+2a^2c^2+b^4-2b^2c^2+c^4$.

25. $a^2-6ab-4ac+9b^2+12bc+4c^2$.

26. $a^4-8a^3b+22a^2b^2-24ab^3+9b^4$.

27. $x^6-8x^5y+16x^4y^2+2x^3y^3-8x^2y^4+y^6$.

28. $p^2+2pq+2pr+2ps+q^2+2qr+2qs+r^2+2rs+s^2$.

29. $a^2+4ab+6ac+2ad+4b^2+12bc+4bd+9c^2+6cd+d^2$.

30. $a^2-4ab-6ac+2ad+4b^2+12bc-4bd+9c^2-6cd+d^2$.

31. $x^6-2x^5+3x^4-x^3+2x+1$.

32. $1-4x+10x^2-14x^3+13x^4-6x^5+x^6$.

33. $1-6x+17x^2-28x^3+28x^4-16x^5+4x^6$.

34. $x^8+2x^7+3x^6-x^4-4x^3-x^2+2x+1$.

35. $a^8-4a^7b-4a^6b^2+12a^5b^3+26a^4b^4+12a^3b^5-4a^2b^6$
$-4ab^7+b^8$.

36. $4x^8-12x^6-4x^5+13x^4+6x^3-5x^2-2x+1$.

37. $a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c$
$+3bc^2+c^3$.

38. $x^3-3x^2y+3x^2z+3xy^2-6xyz+3xz^2-y^3+3y^2z$
$-3yz^2+z^3$.

39. $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$.

40. $27x^6+54x^5y+63x^4y^2+44x^3y^3+21x^2y^4+6xy^5+y^6$.

41. $64x^{12}-96x^{10}+96x^8-56x^6+24x^4-6x^2+1$.

42. $2+18x^2+36x^4+16x^6$.

43. $12a^5x+40a^3x^3+12ax^5$.

44. $a^6-48a^4x^2+768a^2x^4-4096x^6$.

45. $a^{12}-3a^8x^4+3a^4x^8-x^{12}$.

46. $a^9-24a^6x^3+192a^3x^6-512x^9$.

47. $x^{12}+4x^9y^6+6x^6y^{12}+4x^3y^{18}+y^{24}$.

48. $x^4-2x^3-23x^2+24x+144$.

49. $8x^6+12x^5-174x^4-179x^3+1305x^2+675x-3375$.

50. $x^{48}-3x^{32}y^{24}+3x^{16}y^{48}-y^{72}$.

Ex. VI.

1. $2a+2b+3c$.

2. $\frac{11x^3}{6}-\frac{x^2}{12}-\frac{3x}{10}-\frac{5}{4}$.

3. $a^3+\frac{a^2b}{6}-\frac{ab^2}{2}+\frac{b^3}{4}$.

4. $\frac{13a}{12}-\frac{17b}{60}+\frac{11c}{30}$.

5. $11x(a+b)+8y(c+d)$.

6. $4xa-2y(a-b)$.

7. $2bx(a+2b)+2ay(a-2b)$.

8. $a-\frac{2b}{3}-\frac{3d}{2}$.

9. $\frac{x^3}{2}+x^2+x-\frac{1}{3}$.

10. $\frac{3a^4}{5}-\frac{a^3}{3}+a^2-\frac{11a}{4}$.

11. $a(x+y)-b(x-y)$.

12. $3xy+\frac{a}{4}(x-2y)+y$.

13. $3a^2+10ab+7b^2-\frac{c}{2}(a-b)-\frac{4c^2}{3}$.

14. $ay-bx$.

15. $\frac{a^2}{3}-\frac{3b^2}{4}$. 16. $a^2-\frac{a^2b}{2}-\frac{23ab^2}{5}-b^3$.

17. $\frac{x^2}{y^2}+1+\frac{y^2}{x^2}$. 18. $\frac{9x^3}{y^3}+\frac{2x}{y}+\frac{y}{x}$.

19. $\frac{120b^4}{a^8}-\frac{101b^6}{a^7}+\frac{69b^8}{a^6}-\frac{18b^{10}}{a^5}$.

20. $2y(a^2-b^2)+2b(x^2-y^2)$. 21. $b^2-2ab-3a^2$.

22. $5x^2-4x+3$. 23. $x^2+2ax+\frac{a^2}{3}$.

24. $a+1+\frac{1}{a}$. 25. $2\left(a^2+2a+3+\frac{2}{a}+\frac{1}{a^2}\right)$.

26. $x-y-z$. 27. $a^2+b^2+2(c^2+d^2)$.

28. $a^3+(m+1)a^2-(m-1)a+m$.

29. $x(2x^2+4xy+y^2)$. 30. $x-y+z+1$.

31. $x+y+z$. 32. $x^2+(a+b)x-ab$.

33. $(ax+by)(x+y)$. 34. $a-x$.

35. $(a+c)^2-b^2$. 36. $(1+x^4)(1-x^8)$.

37. $a^4+18a^2b^2+9b^4$.

38. $2(a^2+b^2+c^2-bc-ca-ab)$.

39. $x^4-x^3y+x^2y^2-xy^3+y^4$.

40. $x^2+2xy+4y^2$. 41. $x^6+x^5y+x^4y^2+$ .....

42. $1-3x+9x^2$. 43. $x^3+3x^2y+9xy^2+27y^3$.

44. x^5-x^4y+ .....

45. $27x^3-18x^2y+12xy^2-8y^3$.

46. $x^6-x^4y^2+x^2y^4-y^6$. 47. $x^6+5x^3y^2+25y^4$.

48. $81x^8+27x^6y+9x^4y^2+3x^2y^3+y^4$.

49. First by both; second by $x+a$; last by $x-a$.

50. First by $x-1$; second by both; third by $x+1$.

Ex. VII.

1. $a^2b^3c^3$. 2. $6a^4b^3x^5y^3$. 3. $180x^2y^5z^4$.
4. $20k^3l^2x^3yz^3$. 5. $60a^3b^3c^3x^3$.
6. $180a^2b^3c^3x^3y^3z^3$. 7. $6ab\,(a+b)$.
8. $12abc\,(a-b)^2$. 9. $120x^2y^2\,(x+y)^2$.
10. $12x\,(x+y)\,(x^2-y^2)$. 11. $12x^2yz\,(x+y)\,(x+z)$.
12. $(a+b)^2\,(a-b)$. 13. $24a^2b^3\,(a^4-b^4)$.
14. $24x^3\,(a+b)^3$. 15. $ab\,(a^2-b^2)$.
16. $15kl\,(k^4-l^4)$. 17. x^3-y^3.
18. $12x^2\,(x^3+1)$. 19. $2a^3\,(a^3+8)$.
20. $2x^3\,(x^3+27)$. 21. $xy^3\,(8x^6-27y^3)$.
22. $6a^3c\,(a^3+b^3c^3)\,(a^2-b^2c^2)$.
23. $2xy^2z\,(x^6-y^3z^6)\,(x^4-y^2z^4)$.
24. $xy\,(x^4-81y^4)$. 25. $(x+1)\,(x+6)\,(x+5)$.
26. $(x-5)\,(x^2-16)$.
27. $(3x-1)\,(x+7)\,(2x+3)\,(x-4)$.
28. $2\,(2x-1)\,(x+4)\,(5x+3)$.
29. $12xy\,(x^2-4y^2)\,(3x-y)\,(x-y)$.
30. $48a^2b^2c^3\,(b+c)^2\,(a+d)$. 31. $3\,(a^2-b^2)\,(x+y)^2$.
32. $3ax\,(a-1)\,(x+2)^2\,(2x-3)$.

Ex. VIII.

1. $\frac{13x+9y}{6}$. 2. $\frac{17x+a}{15}$. 3. $\frac{7a+9c}{12}$.
4. $\frac{54a-15b+15c}{50}$. 5. $\frac{11x}{12}$. 6. $\frac{3x-7y+26z}{24}$.

7. $\frac{29x-18y}{42}$. 8. $\frac{32-3a}{42}$. 9. $\frac{43x-35}{90}$.

10. $\frac{19y-4x}{12}$. 11. $\frac{5a-27}{18}$. 12. $\frac{29a-60}{30}$.

13. $\frac{5a-14b}{20}$. 14. $\frac{145x-2a}{180}$. 15. $\frac{2a+11x}{12}$.

16. $\frac{y-19a-22x}{60}$. 17. $\frac{7c-3a-18b}{42}$.

18. $\frac{21x^2-51x+10}{36}$. 19. $\frac{76x^2-13x-5}{22}$.

20. $\frac{b^2}{6}$. 21. $\frac{2x}{x^2-y^2}$. 22. $\frac{x^2+y^2}{x^2-y^2}$.

23. $\frac{a^2-2ab-b^2}{a^2-b^2}$. 24. $\frac{k^2+2kl-l^2}{k^2-l^2}$.

25. $\frac{x^2+4a^2}{x^2-4a^2}$. 26. $\frac{a^2+2ab+3b^2}{a^2-9b^2}$.

27. $\frac{x^2+5y}{x^4-y^2}$. 28. $\frac{a^4+x^4}{a^4-x^4}$.

29. $\frac{3k^4+4k^2l^2+4l^4}{k^4-4l^4}$. 30. $\frac{7x^6+12x^3y^3-5y^6}{x^6-y^6}$.

31. $\frac{x^2}{a^2-x^2}$. 32. 0. 33. $\frac{5ab}{a^2-b^2}$.

34. $\frac{a}{a^2-4}$. 35. $\frac{x^4+y^4}{x^4-y^4}$. 36. $\frac{x^2+5}{x^4-9}$.

37. $\frac{2x^4+2ax^3+3ax}{x^6-a^2}$. 38. $\frac{3a^2x}{x^2-9a^4}$.

39. $\frac{x^2-4a}{x^4-4a^2}$. 40. $\frac{a^4+3a^2b}{a^4-9b^2}$. 41. $\frac{2a^2x^2}{x^4-a^4}$.

42. $\frac{8(x+1)}{x^4-16}$. 43. $\frac{2a^2b^2}{a^4-b^4}$. 44. $\frac{54}{a^4-81}$.

45. $\frac{2y^8}{x^8-y^8}$.

46. $\frac{4(8x^4+34x^2+45)}{16x^4-81}$.

47. $\frac{3x}{2(a^2-x^2)}$.

48. $\frac{3x^2+2y^2}{6(x^2-y^2)}$.

49. $\frac{a}{5(9a^2-x^4)}$.

50. $\frac{x^2+16a^2}{6(x^2-16a^2)}$.

51. $\frac{xy^2}{x^3+y^3}$.

52. $\frac{x^2y}{x^3+y^3}$.

53. $\frac{xy^2}{x^3-y^3}$.

54. $\frac{ab^2}{a^3-b^3}$.

55. $\frac{2x^2y^4}{x^6+y^6}$.

56. $\frac{b^6-a^2b^4}{a^6+b^6}$.

57. $-\frac{2a^2b^2}{a^6+b^3}$.

58. $\frac{a^2x^2}{a^3-x^6}$.

59. $\frac{x^2-3x}{x^3+8}$.

60. 0.

61. $\frac{x^2+8}{(x+2)(x-4)}$.

62. $\frac{7ab}{(a-2b)(3a+b)}$.

63. $\frac{15y}{(4x-y)(2x+7y)}$.

64. $\frac{30x^2}{(3x^4-2y^4)(6x^2+y^2)}$.

65. $\frac{2x+9y}{24xy(2x+y)}$.

66. 0.

67. $\frac{2x+3y}{xy(x+y)}$.

68. $\frac{3a^2-2b^2}{ab(a+2b)}$.

69. $\frac{9y-2x}{12xy(x+3y)}$.

70. $\frac{2b^2-3a^2}{ab(a-b)}$.

71. $\frac{2x+3y}{xy(x-y)}$.

72. $\frac{2a^2+3b^2}{ab(a-3b)}$.

73. $\frac{4x+6}{(x+1)(x+2)(x+3)}$.

74. $\frac{x^3+x^2y-9xy^2-6y^3}{(x+y)(x^2-4y^2)}$.

75. $\frac{(7a+b)b^2}{(2a+b)(3a-b)(6a+b)}$.

76. $\frac{29a^2-57ab+4b^2}{(5a-b)(2a-3b)(a+b)}$.

77. $\frac{4x^3+x^2y-3xy^2+4y^3}{6(x+2y)(x^2-y^2)}$.

78. $\dfrac{14x^3-5a^2}{ax(a-x)(a+2x)}$.

79. $\dfrac{47xy-61x^2-132y^2}{6xy(x+y)(x-3y)}$.

80. $\dfrac{ab(a-c)-c^2(a-b)}{a(a-c)(b-c)}$.

81. $\dfrac{2ab-b^2-c^2}{(a-b)(a-c)(b-c)}$.

82. $\dfrac{c}{(a-b)(c-a)(b-c)}$.

83. $\dfrac{y}{(x-y)(y-z)}$.

84. $\dfrac{6kl-5km-lm}{(k-l)(k-m)(l-m)}$.

85. $\dfrac{2a-b-c}{(b-c)(c-a)(a-b)}$.

86. $\dfrac{5x-y-4z}{(y-z)(z-x)(x-y)}$.

87. $\dfrac{3y^2-xy+3yz-5z^2}{(x-y)(y-z)(z-x)}$.

88. $\dfrac{3ab+2ac-5bc}{(a-b)(a-c)(b-c)}$.

89. $\dfrac{2x+5}{(x+2)(x+3)}$.

90. $\dfrac{x-12}{(x+3)(x+4)(x-2)}$.

91. $\dfrac{3(3x+2)}{(x-4)(x+3)^2}$.

92. $\dfrac{3x^2+10x+45}{3(x-9)(x+2)(x+5)}$.

93 $\dfrac{6}{(5-2x)(x+3)(2x+7)}$.

94. $\dfrac{12x^2-7}{2(x-1)(3x-5)(2x-1)}$.

95. $\dfrac{5}{(x+1)(x+2)(x+4)}$.

96. $\dfrac{2x^3-9x+3}{2(x^2-1)(x-2)}$.

97. $\dfrac{2x^3-125x+24}{2(4x^2-9)(x-5)(x+4)}$.

98. $\dfrac{6x^3-13x^2+3x-260}{6(4-x)(2x+1)(x^2-25)}$.

99. $\dfrac{x^4+33x-108}{4x(x^2-9)}$.

100. $\dfrac{82x^2-227x+40}{x(2x-5)(x+5)(2-x)}$.

Ex. IX.

1. $\frac{c^2}{bz}$.
2. $\frac{9c^2}{a^2}$.
3. $\frac{y^2}{2a^2}$.
4. $\frac{ab^2c}{3x^2yz}$.
5. $\frac{d}{8a}$.
6. $\frac{3xy}{a^2}$.
7. $\frac{y(x-y)}{x(x+y)}$.
8. $\frac{a^2-b^2}{a^2}$.
9. $\frac{x}{2x-y}$.
10. $\frac{3(2a-3b)}{2(a-b)}$.
11. $\frac{8(x-3y)z}{5x^2}$.
12. $\frac{2(x^2+y^2)}{y(x+y)}$.
13. $\frac{3(x^2-y^2)(x-y)}{x(x^2+y^2)}$.
14. $\frac{3a(2a^2-b^2)}{(2a+b)(a+b)}$.
15. $\frac{5(9x^4+6x^2y+4y^2)}{3x(x+y)(x^2-y^2)}$.
16. $\frac{x}{4y}$.
17. $\frac{2(x^2-4y^4)}{x(x+y)}$.
18. $\frac{4(a^2-3bc^3)}{3(a^2+ab^2+b^4)}$.
19. $\frac{3a(x+3y)}{2xy}$.
21. $\frac{x+1}{x+2}$.
22. $\frac{x+4}{x-9}$.
23. $\frac{(3x+2)^2}{(x+2)(x-1)}$.
24. $\frac{3(x+7)}{2(2x-1)}$.
25. $\frac{x^3(x+2y)}{2(x-4y)(x+y)}$.
26. $\frac{2(x^2+a^2)(x^2-2ax+4a^2)}{3ax^3}$.
27. $\frac{3(x^2+1)(x^4+1)}{x^3(x^3+1)}$.
28. $\frac{(x+12)(x^2-3)}{x(x+3)(x-9)}$.
29. $\frac{4a(x-2a)}{3x(x+2a)}$.
30. $\frac{9(x+y)(3x^2-y^2)}{x^2(x-y)}$.
31. $\frac{a}{x}$.
32. $\frac{a^4-9x^2}{3a^2}$.
33. $\frac{3x}{x-y}$.

34. x^3. 35. $\frac{1}{3}$. 36. $\frac{a^2b^2}{2(a-b)}$.

37. $\frac{x+y-z}{x-y-z}$. 38. $\frac{2a+b-c}{2a+b+c}$. 39. 1. 40. $\frac{a+2b-3}{a-2b-4}$.

Ex. X.

1. $\frac{a+x}{a}$. 2. $\frac{3x-a}{x-a}$. 3. $\frac{2(x+5y)}{x+2y}$.

4. $\frac{a^2+2ax}{a+x}$. 5. $\frac{2ab+b^2}{a+2b}$. 6. $\frac{x^2-4xy}{x-y}$.

7. $\frac{a^2-4b^2}{a-b}$. 8. $\frac{y^3}{x^2-xy+y^2}$. 9. $x+\frac{5x}{7}$.

10. $2a+\frac{3a}{11}$. 11. $2a^2+\frac{8a^2}{15}$. 12. $4a+\frac{17a}{50}$.

13. $1+\frac{2b}{a}$. 14. $x+y+\frac{y^2}{x}$. 15. $2x-\frac{2y^2}{3x}$.

16. $2a-2b-\frac{b}{2}$. 17. $a+\frac{b^2}{a+b}$. 18. $x-y+\frac{y^2}{x-y}$.

19. $1-\frac{6x}{4a+3x}$. 20. $1-\frac{b+2c}{2a+c}$. 21. $\frac{a+b}{a^2+b^2}$.

22. 1. 23. $\frac{2xy}{x^2+y^2}$. 24. $\frac{x(x^3+2x-1)}{(x+1)(x^2-3x+1)}$.

25. $\frac{4ax}{a^2+x^2}$. 26. $\frac{1}{(x-y)^2}$. 27. 1. 28. 1.

29. $a(x+y-2a)$. 30. $a-1$. 31. $\frac{3b(a+b)}{2a(a-b)}$.

33. $\frac{2x-2}{x}$. 34. $\frac{3}{2x}$. 35. $\frac{3x+a}{x}$.

36. $\frac{a-8x}{a-2x}$. 37. $-\frac{6a^2}{x}$. 38. $-\frac{a}{2}$.

39. $\frac{2xy}{x+3y}$. 40. $\frac{2x+11a}{4a}$. 41. $\frac{7x-5y}{3x-2y}$.

42. $\frac{3y+2}{4}$. 43. $\frac{5}{3x+2}$. 44. $\frac{7x}{6x-3}$.

45. $\frac{5a-x}{2a}$. 46. $\frac{x-y}{2x}$. 47. $\frac{2(x^4-x^2y^2+y^4)}{x+y}$.

48. $\frac{37}{8(x-7)(x+4)}$. 49. $\frac{3(2x^2-3)}{8(1-x)(2x-1)(3x+1)}$.

50. $\frac{1+2bx+b^2}{(1-b^2)(1-x^2)}$.

Ex. XI.

1. 2. 2. 3. 3. 1. 4. 2. 5. -1.

6. -2. 7. 7. 8. -5. 9. $\frac{1}{2}$. 10. $\frac{1}{8}$.

11. 3. 12. 1. 13. -1. 14. -2. 15. 5.

16. 3. 17. 2. 18. -1. 19. $7\frac{1}{2}$. 20. $\frac{1}{2}$.

21. 0. 22. -2. 23. 2. 24. 1. 25. $2\frac{1}{2}$.

26. 5. 27. $\frac{1}{9}$. 28. $\frac{2}{3}$. 29. 5. 30. 6.

31. -1. 32. 2. 33. 3. 34. -2. 35. -3.

36. 3. 37. -2. 38. 5. 39. 0. 40. -7.

41. 8. 42. -2. 43. 5. 44. -1. 45. $-\frac{1}{2}$.

46. $\frac{2}{3}$. 47. 3. 48. 7. 49. 4. 50. -2.

51. 4. 52. -5. 53. 0. 54. 7.

55. -2. 56. -2. 57. $5\frac{1}{2}$. 58. $6\frac{1}{2}$.

59. $-3\frac{1}{2}$. 60. -5. 61. 1. 62. 3. 63. $\frac{2}{3}$.

64. 12. 65. -1. 66. 2. 67. $4\frac{1}{2}$. 68. $\frac{6}{7}$.

69. $-\frac{4}{5}$. 70. 3. 71. 4. 72. 12. 73. 20.

74. 8. 75. -5. 76. 5. 77. 10. 78. 56.

79. 2. 80. 3·1875. 81. 4. 82. 8. 83. 6.

84. 1. 85. 12. 86. 9. 87. 18. 88. 3.

89. 7. 90. 10. 91. 2. 92. 2. 93. 5.

94. $\frac{9}{20}$. 95. 4. 96. $-\frac{23}{13}$. 97. 8. 98. 5.

99. -14. 100. $\frac{1}{2}$.

Ex. XII.

1. 20. 2. 700. 3. 75; 45. 4. 36; 16.

5. 45; 27. 6. £187. 10*s*. 7. 13*s*. 3*d*.; 7*s*. 9*d*.

8. £2. 5*s*. 9. £40. 10. 19*s*. 11. 65; 52.

12. 55. 13. £2050. 14. £1000; £500; £300.

15. £850. 16. £20; £80; £105. 17. 78; 72.

18. $\frac{7}{11}$. 19. 7; 32. 20. 11; 24. 21. $5\frac{5}{8}$ miles.

22. £325. 23. 47. 24. 64. 25. 63 yards.

26. $4\cdot54\frac{6}{11}'$. 27. $2\cdot10\frac{10}{11}'$. 28. $6\cdot16\frac{4}{11}'$; $6\cdot49\frac{1}{11}'$.

29. $3\cdot38\frac{2}{11}'$. 30. $7\cdot16\frac{4}{11}'$. 31. 24 feet. 32. 3 yds.

33. 3; 1. 34. 55 and 45 miles an hour.

35. 18 miles; 18 minutes. 36. £20; £2. 10*s*.

37. 45; 46. 38. $3\frac{1}{2}$ miles an hour. 39. $13\frac{1}{2}$ miles.

40. $775\frac{95}{111}$. 41. $6\frac{3}{4}$. 42. $\frac{3}{11}'$. 43. 72.

44. 7*s*. 6*d*. 45. £1. 46. £1. 2*s*. 6*d*.

47. $\frac{1760ac}{b(c-d)}$ yards. 48. £400; £500.

Ex. XIII.

1. $-\frac{1}{2}$; 5. 2. 1; 3. 3. 5; 4. 4. ·3; 2·1.

5. 6; 2. 6. 10; 5. 7. 1, 0; $\frac{1}{2}$, 0.

8. 4, 0; $2\frac{1}{2}$, 0. 9. 20, 0; 1, 0. 10. 4; 10.

11. 1; 2; 3. 12. 2; 4; 5. 13. ·5; 1; 2.

14. 2; 3; 5. 15. $\frac{1}{2}$; $\frac{1}{3}$; 1. 16. 1; 1; 2.

17. 3; 2; 1. 18. 4; 5; 1; 2. 19. 6; 3; 2; 0.

20. 1; 2; 3; 4. 21. 4; 2; 1; $\frac{1}{2}$.

Ex. XIV.

1. £250; £320. 2. $\frac{9}{20}$. 3. 5. 4. $\frac{5}{24}$.

5. $8\frac{1}{8}$; $1\frac{1}{8}$. 6. $\frac{5}{19}$. 7. 70; 50. 8. 45; 54.

9. 36. 10. 9; 12. 11. $\frac{7}{12}$. 12. 21; 40.

13. 2*s.* 6*d.*; 8*d.* 14. 4 yds.; 5 yds. 15. 253.

16. 90 : 79. 17. 12 ft; 7 ft. 18. 7225.

19. 4, 5; 18, 19. 20. 11; 120 ft. 21. £3.

22. 10 gal.; 12 gal. 23. In proportion of 1 : 2.

24. 6 hr. $52\frac{1}{143}$′; 10 hr. $34\frac{58}{143}$′. 25. 4 miles an hour; 8 miles an hour. 26. 570. 27. £540; £360.

28. £280; £350. 29. 50; 60. 30. 90 : 79.

31. 50.

Ex. XV.

1. $\frac{a^2+b^2}{a+b}$. 2. $\frac{a^2+b^2}{a-b}$. 3. $a-b$. 4. $\frac{a^2+b^2}{a-b}$.

5. $a+2b$. 6. $c-d$. 7. $\frac{2a-c}{2}$. 8. $\frac{a^2-b^2}{a-b+c}$.

9. $a-b+c$. 10. $\frac{2a-b}{2}$. 11. $\frac{ab}{a+b}$.

12. $\frac{2ab}{b-a}$. 13. $a-b$. 14. $a+b$. 15. b.

16. $\frac{c^2+2ac-a^2}{2c}$. 17. $\frac{2ac^2}{(a-b)^2}$. 18. $2a-b$.

19. $\frac{a^2-ab+b^2}{2(b-a)}$. 20. $\frac{(a-b)^2}{a+b}$. 21. $\frac{bc+ca+ab}{2(a+b+c)}$.

22. $\frac{b(2a-3c)}{3a-b-3c}$. 23. $\frac{11a}{2}$. 24. $\frac{11a}{2}$. 25. $\frac{1}{ab}$.

26. $\frac{b(a^2-ab+1)}{a(b^2+b-1)}$. 27. $a-b$. 28. $\frac{b}{a-1}$.

29. $\frac{a^2(b-a)}{b(b+a)}$. 30. a. 31. a^2. 32. $a+b$.

33. $a+2b$. 34. $\frac{4a^2+1}{4}$. 35. $2(c-a-b)$.

36. 0; a. 37. b; 0. 38. $\frac{bc+c^2}{a+b}$; $\frac{c^2-ac}{a+b}$.

39. $\frac{ab}{a+b}$; $\frac{a^2}{a+b}$. 40. $\frac{ac+bd}{a^2+b^2}$; $\frac{ad-bc}{a^2+b^2}$.

41. $-a$; $-b$. 42. $\frac{c^2}{c+d}$; $\frac{d^2}{c+d}$. 43. c; d.

44. 1; 1. 45. $\frac{2ab}{a+b}$; $\frac{2ab}{b-a}$. 46. a; b.

47. $\frac{a^2-b^2}{ac-bd}$; $\frac{a^2-b^2}{ad-bc}$. 48. $\frac{3ac^2}{a^2+c^2}$; $\frac{c(2a^2-c^2)}{a^2+c^2}$.

49. $\frac{5(a+2b)}{6}$; $\frac{11a-14b}{6}$; $\frac{a+2b}{6}$. 50. $\frac{a}{6}$; $\frac{5a}{12}$; $\frac{2a}{3}$.

51. $\frac{2}{23a}(15bc-4ca+4ab)$; &c.

52. $\frac{m+n}{2}$; $\frac{n+l}{2}$; $\frac{l+m}{2}$.

Ex. XVI.

1. 8; -2. 2. 15; 3. 3. 5; -2.

4. 3; $-2\frac{5}{6}$. 5. $\frac{2}{5}$; $\frac{1}{5}$. 6. 1; $-\frac{5}{6}$.

7. 2. 8. 4; $-\frac{1}{2}$. 9. 5; -2.

10. 10; $-\frac{1}{2}$. 11. 4; -2. 12. 1; 10.

13. 8; 3. 14. 5; $2\frac{1}{2}$. 15. 6; $1\frac{1}{2}$.

16. 10; $5\frac{1}{3}$. 17. 1; $-9\frac{1}{2}$. 18. $\frac{5}{6}$; $-\frac{2}{3}$.

19. $3\frac{1}{2}$; $-5\frac{1}{4}$. 20. 2; $\frac{2}{7}$. 21. 5; -2.

22. $8\frac{2}{3}$; -2. 23. 4; $-4\frac{2}{3}$. 24. 3; 0.

25. 5; $4\frac{1}{2}$. 26. -3; -2. 27. 8; $-1\frac{2}{3}$.

28. 3; $1\frac{2}{3}$. 29. 0; $-1\frac{1}{2}$. 30. $\frac{1}{10}$; -1.

31. -7; $-8\frac{1}{8}$. 32. $\frac{2}{3}$; $-\frac{13}{42}$. 33. $2\frac{1}{2}$; 0.

34. $-\frac{1}{3}$; -4. 35. 1; $-\frac{10}{13}$. 36. $-1\frac{5}{7}$; -3.

37. 2; $-2\frac{3}{5}$. 38. 3; -8. 39. 10; $7\frac{1}{3}$.

40. 5; $-1\frac{1}{3}$. 41. 4; $2\frac{15}{23}$. 42. -1; $-1\frac{5}{6}$.

43. $\frac{1}{20}$; -2. 44. 3; $-1\frac{2}{15}$. 45. 4; 2.

46. 4; $-5\frac{8}{11}$. 47. 4; $2\frac{87}{89}$. 48. -3; $-2\frac{17}{34}$.

49. $\frac{1}{2}$; $-\frac{103}{148}$. 50. $\frac{1}{2}$; $-2\frac{4}{113}$. 51. 37; -2.

52. 3; $-8\frac{7}{10}$. 53. 1; 4. 54. 3; $-\frac{1}{2}$.

55. 4. 56. $2\frac{3}{7}$; -1. 57. 2; -7.

58. 3; 1. 59. 5; 2. 60. 2; -1.

61. 5; -8. 62. 5; -6. 63. 5; -2.

64. 4; $-2\frac{1}{2}$; 3; $-1\frac{1}{2}$. 65. 5; $-2\frac{2}{3}$.

66. 1; -9. 67. 4. 68. 17. 69. 11.

70. 7. 71. 8. 72. 6. 73. ± 2.

74. ± 3. 75. $\pm a\sqrt{\frac{a-b}{a+b}}$. 76. ± 6.

77. $a; \frac{a}{5}$. 78. $\pm\sqrt{2}$. 79. $\pm\frac{m}{2}\sqrt{3}$.

80. $3; -1$. 81. 5. 82. $1; -3\frac{4}{15}$. 83. $2; -4\frac{3}{4}$.

84. ± 2. 85. 2. 86. $4; 1$. 87. 9.

88. $32; -243$. 89. $3; 2$. 90. $1; \frac{1}{9}$.

91. $1; \frac{1}{16}$. 92. $\frac{a}{2}; \frac{3a}{4}$. 93. $0; \frac{a(m+n-2p)}{mp+np-2mn}$.

94. $0; -\frac{2(a^2+ab+b^2)}{a+b}$. 95. $-a; b$. 96. 3.

97. 2. 98. 2. 99. 3. 100. 4. 101. $3; 1\frac{1}{4}$.

102. $3; -3\frac{1}{3}$. 103. $\pm 2; -1$. 104. $2; -1\frac{1}{2}$.

105. $3; -\frac{4}{5}$. 106. 4. 107. $3; -1; 1\pm\sqrt{3}$.

108. $0; 1; -9$. 109. $6; 2; -4$.

110. $1; 2; \frac{3\pm\sqrt{13}}{2}$.

Ex. XVII.

1. $x^2-6x+8=0$.
2. $x^2-16x+63=0$.
3. $2x^2-17x+8=0$.
4. $3x^2-304x+400=0$.
5. $x^2+9x+14=0$.
6. $x^2+9x+8=0$.
7. $5x^2+8x=4$.
8. $5x^2+76x+15=0$.

9. $x^3-8x^2+19x-12=0$. 10. $x^3-3x^2-4x+12=0$.

11. $2x^3-15x^2+4x+21=0$. 12. $24x^3-38x^2+15x=0$.

13. $x^3-2x^2-8x=0$. 14. $15x^3+76x^2-x=30$.

15. $x^4-4x^3-4x^2+16x=0$. 16. $x^4-7x^3-x^2+7x=0$.

17. $\frac{b^2}{a}$; $-ka$. 18. $\frac{124}{25}$; $\frac{2\sqrt{26}}{5}$.

19. $\frac{b^2-2ac}{a^2}$; $\frac{3abc-b^3}{a^3}$. 20. $cx^2+bx+a=0$.

21. $a^4x^2-2a^3cx-b^4+6ab^2c-8a^2c^2=0$.

Ex. XVIII.

1. 9; 1. 2. 2; 1. 3. 2; 1. 4. 5; 0.

5. 5; 1. 6. 7; −2. 7. 3; 1. 8. 2; −1.

9. ±6, ±5; ±5, ±6. 10. 9, 8; 8, 9.

11. 9, −3; 3, −9. 12. 4, $-\frac{2}{3}$; 2, −12.

13. 3, $\frac{1}{4}$; 1, 12. 14. 6, $-\frac{4}{5}$; 2, −15.

15. 4, $-1\frac{3}{7}$; 1, $-2\frac{1}{3}$. 16. 4, 3; 3, 4.

17. 2, 3; 3, 2. 18. 5, −3; 3, −5.

19. 7, −1; 1, −7. 20. 8, −6; 3, −4.

21. 4, −3; 1, $-\frac{4}{3}$. 22. 6, −4; 2, −3.

23. 3, $1\frac{4}{5}$; 1, $2\frac{1}{5}$. 24. 2, $-\frac{18}{13}$; 1, $\frac{35}{13}$.

25. 2, $1\frac{7}{11}$; 1, $-\frac{1}{11}$. 26. 5, −6; −2, $5\frac{1}{3}$.

27. $2, \frac{5}{8}; \ -7, \ -\frac{1}{8}.$

28. $3, \ 1\frac{1}{3}; \ 2, \ 1\frac{1}{3}.$

29. $13, \ 67; \ 7, \ -47.$

30. $3, \ -1\frac{20}{27}; \ -4, \ 8\frac{1}{27}.$

31. $3, \frac{3}{13}; \ 2, \ 3\frac{1}{13}.$

32. $5, \ \pm 2; \ -5\frac{1}{4}, \ \frac{\sqrt{105}}{4}.$

33. $10, \ 11\frac{8}{5}; \ 12, \ 9\frac{3}{5}.$

34. $7, \ -13\frac{18}{25}; \ 6, \ -29\frac{18}{25}.$

35. $\pm 7; \ \pm 1.$

36. $2, \ 1; \ 1, \ 2.$

37. $5, \ 1; \ 1, \ 5.$

38. $5, \ -2; \ 2, \ -5.$

39. $6, \ -2; \ 2, \ -6.$

40. $3, \ 4; \ 2, \frac{3}{2}.$

41. $1, \ -\frac{2}{3}; \ 2, \ -3.$

42. $3, \ -2\frac{1}{2}; \ 1, \ -1\frac{1}{5}.$

43. $4, \ -2; \ 3, \ -6.$

44. $3, \ 2; \ 2; \ 3.$

45. $5, \ 1; \ 1, \ 5.$

46. $3, \ 2; \ -2, \ -3.$

47. $1, \frac{1}{2}; \ -1, \ -2.$

48. $1, \ 1\frac{1}{3}; \ 2, \ 1\frac{1}{2}.$

49. $1, \ 2; \ 2, \ 1.$

50. $3, \ -1; \ 1, \ -3.$

51. $3, \ -2; \ 1, -\frac{3}{2}.$

52. $-1, \ 1\frac{1}{3}; \ 2, \ -1\frac{1}{2}.$

53. $4, -3; \ 1, \ -1\frac{1}{3}.$

54. $1\frac{1}{2}, \ -1; \ 1, \ -1\frac{1}{2}.$

55. $\pm 4; \ \pm 1.$

56. $\pm 3; \ \pm 2.$

57. $\pm 5, \ \pm 16; \ \pm 1, \ \mp 4\frac{1}{2}.$

58. $\pm 2, \ \pm 2\frac{4}{5}; \ \pm 4, \ \pm 2\frac{3}{5}.$

59. $\pm 4, \ \pm 5; \ \pm 3, \ \pm 1\frac{1}{2}.$

60. $\pm 3, \ \pm 10\frac{1}{2}; \ \mp 1, \ \pm 4.$

61. $4, \ 2; \ 2, \ 4.$

62. $6, \frac{3}{2}(\sqrt{5}-1); \ 6, \ -\frac{3}{2}(\sqrt{5}+1).$

63. $1, \ -\frac{3}{4}; \ \frac{2}{5}, \ 1\frac{1}{10}.$

64. $10, \ -82; \ \frac{1}{2}, \ 23\frac{1}{2}.$

65. $2, \frac{2}{3}; 1, \frac{1}{3}$. 66. $2, -4; 1, 2\frac{1}{2}$.

67. $2a, 2b, a-b; 2b, -2b, b-a$.

68. $\frac{1}{3}(\sqrt{2a+c}+\sqrt{a-4c}), \frac{1}{3}(\sqrt{2a+c}\pm\sqrt{c-a})$;

$\frac{1}{3}(\sqrt{2a+c}-\sqrt{a-4c}), \frac{1}{3}(\sqrt{2a+c}\pm\sqrt{c-a})$;

$\frac{1}{3}\sqrt{2a+c}, \frac{1}{3}(\sqrt{2a+c}\mp 2\sqrt{c-a})$.

69. 3, 1, 10, 1; 1, 3, 1, 10. 70. $\pm 1; \pm 1$.

71. $\pm 1, 5\sqrt{\frac{5}{26}}; \mp 2, 8\sqrt{\frac{2}{65}}$.

72. $\pm 2, \pm 18\sqrt{\frac{3}{59}}; \pm 3, \mp\frac{11}{3}\sqrt{\frac{3}{59}}$.

73. $\pm 7, \pm\sqrt{3}; \pm 2, \mp 3\sqrt{3}$.

74. $\pm 2, \pm\frac{145}{6}\sqrt{\frac{1}{15}}; \pm 3, \mp\frac{26}{3}\sqrt{\frac{1}{15}}$.

75. $\pm 7, \pm 4\sqrt{\frac{1}{19}}; \pm 1, \mp\frac{25}{2}\sqrt{\frac{1}{19}}$.

76. $\pm 2, \pm\frac{35}{\sqrt{163}}; \mp 3, \pm\frac{43}{\sqrt{163}}$.

77. $\pm 1, \pm\sqrt{\frac{1}{17}}; \mp 3, \mp\frac{11}{\sqrt{17}}$.

78. $\pm 2, 8\sqrt{\frac{2}{57}}; \mp 2, 2\sqrt{\frac{2}{57}}$.

79. $\pm 3, \pm 2; \pm 3, \pm 2$. 80. $5; 1$.

81. 4, $-3\frac{1}{3}$; 2, $-5\frac{1}{3}$. 82. 3, $4\frac{1}{3}$; 4, $1\frac{1}{3}$.

83. 9, 3; 3, 9. 84. ± 3, ± 2; ± 2, ± 3.

85. 4, -1, $\pm\sqrt{5}-1$; 1, -4, $\pm\sqrt{5}+1$.

86. 2, 1, -1, 2; 1, 2, 2, -1.

87. $x=\dfrac{\sqrt{a^2+b^2}+\sqrt{a^2+c^2}-\sqrt{b^2+c^2}}{2}$; &c.

88. $x=a^2\sqrt{\dfrac{b^2+c^2-a^2}{(a^2+b^2-c^2)(a^2+c^2-b^2)}}$; &c.

Ex. XIX.

1. 7 or $1\frac{1}{2}$. 2. 60; 12. 3. 729. 4. 16; 10.
5. 18; 14. 6. 24; 1. 7. 9; 5. 8. 7; 3.
9. 10; 11. 10. 3. 11. 26. 12. 23.
13. 6 ft. 14. 11 ft. 15. 6 yds.; 1 yd. 16. 53.
17. 4 yds.; 5 yds. 18. $3\frac{3}{4}$ hours; $2\frac{1}{2}$ hours.
19. 16 miles; 4 mi. an hr. 20. 6 hrs.; 10 hrs.
21. 18 yds.; 14 yds. 22. £2. 23. £3. 10*s*.
24. £2; £2. 10*s*. 25. 40*s*. 26. 16.
27. 16*d*. a dozen. 28. 10*d*. 29. $33\frac{1}{3}$ per cent.
30. 10 per cent. 31. 4. 32. 4; 5. 33. 188.
34. $4\frac{1}{2}$ miles an hour. 35. $4\frac{1}{2}$ per cent. 36. £90.

Ex. XX.

1. $a+x$. 2. $2x-1$. 3. $3a^2-2x$. 4. $4x+y^2$.
5. $5a^3-2b$. 6. $12x^4-10xy^3$. 7. $1-2x+x^2$.
8. $2x^2-3x+1$. 9. $2a^4-3a^2+4$. 10. $2-4ax^2+a^2x^4$.
11. $3x^2-xy-2y^2$. 12. $4a^4+5a^2b-3b^2$.

13. $2a-1+2a^{-1}$. 14. $x^{\frac{1}{2}}+2x^{\frac{1}{2}}y^{-1}-3y^{-2}$.

15. $a^{\frac{3}{2}}-a^{\frac{1}{2}}b^{-1}-2b^{-2}$. 16. $3x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y^{\frac{1}{2}}$.

17. a^2-2x^2. 18. $3a+x$. 19. $3a^2+\frac{1}{2}x$.

20. $x+y$. 21. 24. 22. 36. 23. 44.

24. 81. 25. 78. 26. 87. 27. 93.

28. 56. 29. 107. 30. 125. 31. 139.

32. 114. 33. 17·9. 34. 245. 35. 2·82.

36. 3·05. 37. 288. 38. 35·6. 39. 709.

40. 6450. 41. 1806. 42. 52·07.

Ex. XXI.

1. 9 : 10, 13 : 15, 4 : 5, 2 : 3.

2. 11 : 12, 17 : 20, 13 : 18, 17 : 10.

3. 61 : 70, 5 : 6, 4 : 7, 2 : 5.

4. The first is greater. 5. 5 : 21. 6. 3 : 40.

7. $b : a-2b$. 8. $x^2-xy : 6y^2$. 9. 4 : 13.

10. 19 : 16. 11. 7. 12. 8. 13. 4.

14. 12; 16. 15. 16; 18. 16. 12; 18. 17. 7.

20. 15; 20. 21. 20; 32. 22. 8; 10.

23. 15; 27. 24. 20; 24. 25. 7; 14.

26. 8. 27. 4.

Ex. XXII.

1. 90. 2. $52\frac{1}{2}$. 3. $17\frac{1}{2}$. 4. $5\frac{1}{3}$. 5. 4.

6. $\frac{a^3}{b^2}$. 7. 2; 0. 8. $\frac{a(c-b)}{a+b}$. 9. 51.

10. 4. 15. 8; 12; 18. 23. 9 : 12 :: 15 : 20.
24. 2. 25. $3\frac{3}{4}$; 5.

Ex. XXIII.

1. $\frac{8B}{5}$. 2. 15. 3. $4B$. 4. 3 : 2.
5. 9. 6. $3\frac{1}{2}$. 7. 9. 8. 6. 9. 3.
10. $4ac^2$. 11. 1. 12. $1+2y$. 13. $3-2y+y^2$.
14. $4+2y+3y^2$. 15. $4x+3x^2$. 16. $23-5B^2$.
18. 72. 19. 96. 20. 19. 21. 30 miles.
22. 12. 23. 12. 24. 15.

Ex. XXIV.

1. $\sqrt{a}+\sqrt[4]{a^3}+4\sqrt[6]{a^5}$. 2. $\sqrt[4]{a^3b}+2\sqrt[6]{a^5b}+3a\sqrt[9]{b^2}$.
3. $\sqrt[4]{x^2y}+3\sqrt[6]{x^2y^5}-2\sqrt[10]{x^2y^3}$. 4. $x^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+2x^{\frac{1}{2}}y^{\frac{2}{3}}$.
5. $a^{\frac{1}{3}}b^{\frac{1}{6}}+2a^{\frac{2}{3}}b^{\frac{2}{6}}-3a^{\frac{1}{2}}b^{\frac{3}{6}}$. 6. $4a^{\frac{1}{2}}b^{\frac{3}{4}}-3a^{\frac{1}{4}}b^{\frac{1}{4}}+4a^{\frac{1}{2}}b^{\frac{7}{12}}$.
7. $a^2b^{-2}+3ab^{-3}c^2-2a^{-2}b^3c^{-1}$.
8. $3xy^{-3}z^{-1}-4x^{-3}y^2z^{-2}+x^3yz^{-4}$.
9. $3ax^{-\frac{1}{2}}+5a^{-2}x^{-\frac{3}{2}}-6ab^{-1}x^{-\frac{1}{2}}y^{-\frac{3}{2}}$.
10. $4x^{\frac{1}{2}}y^{-\frac{3}{2}}-2x^{\frac{1}{2}}y^{\frac{3}{2}}z^{-\frac{3}{2}}-3x^{\frac{1}{2}}y^{-1}z^{-\frac{1}{2}}$.
11. $\frac{b}{a^2c^3}+\frac{2b^3}{ac^3}-\frac{4b}{a^3}$. 12. $\frac{5\sqrt{a}}{b^2}-\frac{7\sqrt{a^3}}{b}+\frac{12\sqrt[4]{a^3}}{\sqrt{b}}$.
13. $\frac{6z^2}{\sqrt[3]{x}\,.\,y}-\frac{8\sqrt{y}}{\sqrt[4]{x^3}}+\frac{\sqrt[3]{y}}{x^2\sqrt[4]{z}}$.
14. $\frac{3\sqrt{yz}}{x^2}-\frac{4z}{y^2\sqrt[3]{x}}+\frac{\sqrt[4]{y^3z}}{2x}$.

15. 4. 16. 9. 17. 27. 18. 27. 19. 25.

20. $32a^{10}$. 21. $9x^4$. 22. $216x^3y^6z^9$. 23. $4x^2y^4z^8$.

24. $\frac{1}{5}$. 25. $\frac{1}{12}$. 26. $\frac{1}{4}$. 27. $\frac{1}{5}$.

28. $\frac{1}{2401}$. 29. $\frac{1}{6a^2}$. 30. $\frac{b^3}{10a}$. 31. $\frac{x^3}{8y^6}$.

32. $\frac{\sqrt[5]{z^3}}{x^2y^4}$. 33. $\frac{x^{15}y^{20}}{z^6}$. 34. $x^{18}\sqrt[3]{y^{10}}$. 35. 1.

36. $\sqrt{y}$. 37. $\frac{x^3z^3}{y^3}$. 38. 3. 39. $\frac{b}{ac^5}$.

40. 12. 41. $\sqrt[24]{a^{11}b^{13}}$. 42. $\frac{2}{3}$.

43. $x^{-2}a^4b^8 - x^{-1}a^2b^4 + 16b^2$.

44. $x^{\frac{1}{2}}(4x^{\frac{1}{2}} + y^{\frac{1}{2}})(2x^{\frac{1}{2}} - 3y^{\frac{1}{2}})$. 45. $xy^{-1} + 1 - x^{-1}y$.

Ex. XXV.

1. $6\sqrt{2}$. 2. $4\sqrt{5}$. 3. $7\sqrt{7}$. 4. $10\sqrt{10}$.

5. $5a\sqrt{3a}$. 6. $8x^2y\sqrt{6x}$. 7. $2\sqrt[3]{2}$.

8. $3\sqrt[5]{5}$. 9. $12x^2y\sqrt{2y}$. 10. $3xy^3\sqrt{2x}$.

11. $20\sqrt{2}$. 12. $28ab\sqrt{2a}$. 13. $60a^2b^2\sqrt{3a}$.

14. $55ax^2\sqrt{5ax}$. 15. $50\sqrt[3]{3}$. 16. $15\sqrt[4]{2}$.

17. $20\sqrt{2}$. 18. 0. 19. $7\sqrt{3}$.

20. $30\sqrt{5}$. 21. $\sqrt[3]{4}$. 22. 0. 23. $6\sqrt{15}$.

24. $70\sqrt{3}$. 25. $210\sqrt{6}$. 26. $47 - 5\sqrt{6}$.

27. $38 + 11\sqrt{15}$. 28. $126 - 10\sqrt{30}$.

29. $12 + 20\sqrt{3}$. 30. $23\sqrt{2} - 7\sqrt{15}$.

31. $54+28\sqrt{10}-5\sqrt{15}+7\sqrt{6}$.

32. $16\sqrt{3}+5\sqrt{6}-12\sqrt{2}-10$.

33. $\frac{75\sqrt{2}}{4}-\frac{8\sqrt{10}-4\sqrt{5}}{5}-5$.

34. $\frac{94\sqrt{2}-23\sqrt{15}}{24}$.

35. $5+\sqrt[3]{36}+\sqrt[3]{6}$.

36. $12-8\sqrt[3]{6}+\sqrt[3]{36}$.

37. $\frac{4\sqrt{3}}{3}$.

38. $\frac{3\sqrt{10}}{5}$.

39. $3+\sqrt{5}$.

40. $\frac{2(4-\sqrt{7})}{3}$.

41. $\frac{17+7\sqrt{5}}{4}$.

42. $\frac{29\sqrt{2}-40}{2}$.

43. $\frac{19+9\sqrt{5}}{8}$.

44. $\frac{12+4\sqrt{5}+3\sqrt{10}+5\sqrt{2}}{6}$.

45. $14+6\sqrt{6}$.

46. $\frac{8\sqrt{2}+\sqrt{10}}{2}$.

47. $\frac{4+5\sqrt{2}+\sqrt{30}}{2}$.

48. $\frac{3\sqrt{30}+4\sqrt{5}+2\sqrt{6}-14}{37}$.

49. $14\frac{1}{2}$.

50. $28\frac{8}{9}$.

51. $\sqrt{10}-\sqrt{3}$.

52. $11+5\sqrt{5}$.

53. $16+11\sqrt{2}$.

54. $\frac{178-43\sqrt{6}}{145}$.

55. $23+6\sqrt{15}$.

56. $\frac{26-15\sqrt{3}+3\sqrt{10}}{13}$.

57. $4+\sqrt{3}$.

58. $6-\sqrt{5}$.

59. $2+\sqrt{10}$.

60. $2\sqrt{5}-\sqrt{2}$.

61. $5+2\sqrt{6}$.

62. $6-2\sqrt{2}$.

63. $\sqrt{7}-\sqrt{3}$.

64. $2\sqrt{5}-\sqrt{11}$.

65. $3\sqrt{10}-2\sqrt{5}$.

66. $4\sqrt{3}-3\sqrt{2}$. 67. $\sqrt{2}-1$. 68. $2+\sqrt{2}$.

69. $\frac{7-\sqrt{2}}{47}$. 70. $7+4\sqrt{3}$. 71. $\frac{\sqrt{5}}{5}$.

72. $4+\frac{3\sqrt{6}}{2}$. 74. $2+\sqrt{3}$.

Ex. XXVI.

1. 28; 61. 2. 62; 86. 3. 19; 40.
4. -25; -41. 5. -11; -149.
6. 81; 861. 7. 42; 315. 8. 307; 15850.
9. 19; $149\frac{1}{2}$. 10. 77; $2014\frac{1}{2}$. 11. 37; 1843.
12. 41; 1127. 13. -102; -782.
14. -355; $-16917\frac{1}{2}$. 15. -90; $-1662\frac{1}{2}$.
16. -11; -55. 17. $18\frac{1}{2}$, 20, $21\frac{1}{2}$, 23, $24\frac{1}{2}$.
18. 5, 3, 1, -1. 19. $3\frac{1}{4}$, 2, $\frac{3}{4}$, $-\frac{1}{2}$, $-1\frac{3}{4}$.
20. 0, $-7\frac{3}{4}$, $-15\frac{1}{2}$. 21. $-\frac{1}{4}$, $1\frac{1}{2}$.
22. $\frac{2}{5}$, $\frac{1}{15}$, $-\frac{4}{15}$. 23. 79. 24. 8.
25. 5, 8, 11..... 26. -18. 27. 41. 28. 13.
29. -2, 2, 6..... 30. 26. 31. 2, $3\frac{1}{2}$, 5...
32. 7, 4, 1..... 33. 8; $-1\frac{1}{2}$.
34. $\frac{3}{4}$, 2, $3\frac{1}{4}$, $4\frac{1}{2}$. 35. 220. 36. $5(a+b)$.
37. $23\frac{7}{9}$, $23\frac{8}{9}$, 24..... 38. $-2n$.
39. *A*, by an hour.

Ex. XXVII.

1. 384. 2. 486. 3. 810. 4. -640.

5. $\frac{1}{54}$. 6. $\frac{1}{160}$. 7. 1024; 2040.

8. 2916; 2188. 9. $\frac{1}{243}$; $1\frac{121}{243}$. 10. $\frac{1}{128}$; $2\frac{85}{128}$.

11. $\frac{1}{243}$; $2\frac{61}{243}$. 12. $\frac{25}{32}$; $2\frac{49}{160}$. 13. 2550.

14. $11\frac{29}{32}$. 15. 2188. 16. 2. 17. $\frac{8}{9}$.

18. $2\frac{5}{11}$. 19. $\frac{9}{20}$. 20. $\frac{64}{195}$. 21. 5, $6\frac{1}{4}$.

22. -2, 1, $-\frac{1}{2}$, $\frac{1}{4}$.

23. $3\frac{29}{32}$, $3\frac{1}{8}$, $2\frac{1}{2}$; 60, 100, $166\frac{2}{3}$.....

24. $\frac{59}{165}$; $\frac{178}{8325}$; $2\frac{229}{3696}$; $32\frac{87}{1815}$; $5\frac{1343}{1998}$.

25. 8. 26. $\frac{1}{3}$.

27. 6. 28. 1, $1\frac{1}{2}$..... 29. 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$.

30. 8, $\frac{8}{3}$, $\frac{8}{9}$..... 31. 49; 1. 34. $\frac{3}{2}$, $\frac{1}{2}$, $\frac{1}{6}$.

36. 10 miles an hour.

Ex. XXVIII.

1. $\frac{5}{9}$, $\frac{5}{11}$, $\frac{5}{13}$. 2. $\frac{10}{17}$, $\frac{10}{21}$, $\frac{2}{5}$.

3. 2, 1, $\frac{2}{3}$..... $\frac{2}{7}$, $\frac{1}{4}$, $\frac{2}{9}$. 4. $\frac{2}{3}$, $\frac{1}{3}$..... $\frac{1}{9}$, $\frac{2}{21}$.

5. 3, $2\frac{1}{4}$, $1\frac{4}{5}$. 6. $\frac{3}{4}$, $\frac{12}{13}$, $1\frac{1}{5}$, $1\frac{5}{12}$; $1\frac{7}{8}$, $1\frac{4}{11}$, $1\frac{1}{14}$, $\frac{15}{17}$.

7. 9, 4. 8. 98, 18. 9. 10, $2\frac{1}{2}$. 10. 4, 9.

11. $1\frac{1}{3}$. 12. $\frac{3}{2}$, $\sqrt{2}$, $\frac{4}{3}$, $\frac{8\sqrt{2}}{9}$, $\frac{32}{27}$.

Ex. XXIX.

1. 330534; 1511425. 2. 163640; 86581.
3. 11111101; 120002. 4. 17187.
5. 5332205; 265323. 6. 75877; 44033.
7. 53341212; 554255112. 8. 2569*te*76; 2*t*346744.
9. 562767452. 10. 3405; 3*e*02.
11. 3014; *t*2*e*. 12. 2043; 120120.
13. 5; 7. 14. 8. 15. 8; 3. 16. 10.
17. 5706. 18. *e*072. 19. 20102. 20. 5732.
21. 32*et*. 22. 3604. 23. 301102. 24. 3475.

Ex. XXX.

1. 32760; 95040. 3. Equal. 4. 360.
5. 2520; 360. 6. 6720. 7. 24; 5040.
8. 665280. 9. 10. 10. 8. 11. 6. 12. 8.
13. 1680; 180; 75600; 75600; 3024. 14. 210; 325.
15. 24; 64. 16. 720. 17. 3360; 1260.
18. 15504; 9880. 20. 190. 21. 495; 165.
22. 1485. 23. $\frac{|48}{|15\,.\,|33}$. 24. 280.

Ex. XXXI.

1. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.
2. $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$.
3. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.
4. $a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+\ldots\ldots$
5. $1-6x+15x^2-20x^3+\ldots\ldots$
6. $1-5a^2+10a^4-10a^6+5a^8-a^{10}$.
7. $a^5+10a^4x+40a^3x^2+80a^2x^3+80ax^4+32x^5$.
8. $x^6+18x^5y+135x^4y^2+540x^3y^3+1215x^2y^4+1458xy^5+729y^6$.
9. $a^{16}-8a^{14}x+28a^{12}x^2-56a^{10}x^3+70a^8x^4-56a^6x^5+28a^4x^6-8a^2x^7+x^8$.
10. $32-240x+720x^2-1080x^3+810x^4-243x^5$.
11. $64a^6-96a^5b+60a^4b^2-20a^3b^3+\frac{15a^2b^4}{4}-\frac{3ab^5}{8}+\frac{b^6}{64}$.
12. $32-60x+45x^2-\frac{135x^3}{8}+\frac{405x^4}{128}-\frac{243x^5}{1024}$.
13. $x^7-21x^6+189x^5-945x^4\ldots\ldots$
14. $x^{13}+52x^{12}y+1248x^{11}y^2+18304x^{10}y^3+\ldots\ldots$
15. $64-960a+6000a^2-20000a^3+\ldots\ldots$
16. $x^{20}-20x^{19}y+190x^{18}y^2-1140x^{17}y^3+\ldots\ldots$
17. $6561+5832a+2268a^2+504a^3+\ldots\ldots$
18. $512-6912x+41472x^2-145152x^3+\ldots\ldots$
19. $1-x-3x^2-13x^3-65x^4\ldots\ldots$
20. $1-6x^2+6x^4+4x^6+6x^8+\ldots\ldots$
21. $32-60a+\frac{135a^2}{4}-\frac{135a^3}{32}-\frac{405a^4}{1024}+\ldots\ldots$
22. $16-8x+\frac{x^2}{2}+\frac{x^3}{24}+\frac{5x^4}{768}+\ldots\ldots$

23. $1+3x+9x^2+27x^3+81x^4+\ldots\ldots$

24. $1+4x^2+16x^4+64x^6+256x^8+\ldots\ldots$

25. $1-2a^2+3a^4-4a^6+5a^8+\ldots\ldots$

26. $1+6a+24a^2+80a^3+240a^4+\ldots\ldots$

27. $1+14x^2+112x^4+672x^6+3360x^8+\ldots\ldots$

28. $a^{-2}+6a^{-3}x+27a^{-4}x^2+108a^{-5}x^3+405a^{-6}x^4\ldots\ldots$

29. $a^{-4}-8a^{-5}x+40a^{-6}x^2-160a^{-7}x^3+560a^{-8}x^4\ldots\ldots$

30. $x^{-6}-2x^{-7}a+\dfrac{7x^{-8}a^2}{3}-\dfrac{56x^{-9}a^3}{27}+\dfrac{14x^{-10}a^4}{9}+\ldots\ldots$

31. $x^{-6}+3ax^{-8}+6a^2x^{-10}+10a^3x^{-12}+15a^4x^{-14}+\ldots\ldots$

32. $a^{-6}+6x^2a^{-8}+24x^4a^{-10}+80x^6a^{-12}+240x^8a^{-14}+\ldots\ldots$

33. $x^4+\dfrac{8}{3}xy^3+\dfrac{8}{9}x^{-2}y^6-\dfrac{32}{81}x^{-5}y^9+\dfrac{80}{243}x^{-8}y^{12}+\ldots\ldots$

34. $1-x+\dfrac{x^2}{6}+\dfrac{x^3}{54}+\dfrac{x^4}{216}+\ldots\ldots$

35. $1+\dfrac{2x}{3}+\dfrac{8x^2}{9}+\dfrac{112x^3}{81}+\dfrac{560x^4}{243}+\ldots\ldots$

36. $1-2x+5x^2-\dfrac{40x^3}{3}+\dfrac{110x^4}{3}+\ldots\ldots$

37. $a^{-\frac{1}{3}}+a^{-\frac{4}{3}}b+2a^{-\frac{7}{3}}b^2+\dfrac{14a^{-\frac{10}{3}}b^3}{3}+\dfrac{35a^{-\frac{13}{3}}b^4}{3}+\ldots\ldots$

38. $a^{-5}-10x^2a^{-7}+70x^4a^{-9}-420x^6a^{-11}+2310x^8a^{-13}+\ldots\ldots$

39. $\dfrac{1}{8}\left\{1+\dfrac{3x^2a^{-5}}{4}+\dfrac{15x^4a^{-7}}{32}+\dfrac{35x^6a^{-9}}{128}+\dfrac{315x^8a^{-11}}{2048}+\ldots\ldots\right\}$.

40. $\dfrac{1}{4}\left\{a^{-2}+\dfrac{xa^{-5}}{3}+\dfrac{5x^2a^{-8}}{36}+\dfrac{5x^3a^{-11}}{81}+\dfrac{55x^4a^{-14}}{1944}+\ldots\right\}$.

41. $\dfrac{32y^5}{243}$.

42. $-\dfrac{6237}{2048}$; $\dfrac{63}{8}$; $-\dfrac{2618}{15625}$.

Ex. XXXII.

1. 10. 2. $7a+4c$.

3. $(x+12)(x-11)$; $(x+5)(x^2-5x+25)$; $(x-3)(x+3)(x^2+9)$.

4. $(3x+2)(2x-1)(3x^2+14x+8)$.

5. $\frac{1}{x^2-a^2}$; $\frac{4(a^4+b^4)}{a^4-b^4}$; $\frac{2x^2}{x^3+y^3}$; $\frac{x^2y^2}{x^2-y^2}$. 7. 3; 4; -3.

8. (i) 3; $1\frac{1}{3}$. (ii) 7, -5; 5, -7. 9. 974.

10. ·192. 11. 100. 12. $a^2+2ab+b^2-c^2$.

13. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$; $(x-4)(x+3)$; $x(2x+3)(3x-1)$.

14. x^2-y^2; $(x^2+y^2)(x^6-y^6)$. 15. $\frac{4ax(a^2-bx)}{x^4-a^4}$.

16. $x^2-x+2-\frac{1}{x}+\frac{1}{x^2}$.

17. (i) 2, -3. (ii) ·5; $\frac{32}{11}$. (iii) ± 3, 0; ± 2, $\sqrt{34}$.

18. 8 hr. $27\frac{3}{11}$ min.

19. $a^{\frac{2}{3}}b^{-\frac{2}{3}}-2a^{-\frac{1}{3}}b^{\frac{1}{3}}+a^{-\frac{2}{3}}b^{\frac{2}{3}}-2a^{\frac{1}{3}}b^{-\frac{1}{3}}+3$. 20. a; $\frac{a}{2}$.

21. $4x^2-\frac{xy}{3}+\frac{y^2}{9}$. 22. $x-1$. 23. $\frac{4a^2}{a^2-b^2}$.

24. (i) $1\frac{1}{2}$. (ii) 4; 3; 2. (iii) $\frac{1}{4}$; -2. (iv) 7, 2; 2, 7.

25. £37. 10s. 26. $5x^3-26x^2+5x=0$.

27. 1; $a^2b^{-\frac{1}{2}}-1+a^{-2}b^{\frac{1}{2}}$. 28. $3-\sqrt{3}$. 29. 20.

31. $a^{-2}+a^{-1}b^{-1}+b^{-2}$.

32. (i) $\frac{1}{4}$; $\frac{1}{5}$. (ii) 4, -2; 2, -4. 33. £25.

35. $\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$. 36. (i) $\frac{bf(g-h)}{bfq-af-2bc}$. (ii) 10; $-1\frac{1}{2}$.

37. $\frac{(a^2+b^2)^2}{a^4+b^4}$. 38. 3, 7, 11,.....

39. $(a^m b)^{\frac{1}{m+1}}$, $(a^{m-1}b^2)^{\frac{1}{m+1}}$,..... 40. 7531.

41. $18x^3-5x^2-2x-11$. 42. $\frac{x^2+3x+2}{2x^2+5x-3}$.

43. (i) 4; 5; 8. (ii) 13; $-4\frac{1}{3}$. 44. $x^{\frac{4}{3}}-3a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}$.

45. 21, 24. 46. 57.

47. (i) 69. (ii) $\frac{8}{3}\left\{\left(\frac{5}{4}\right)^n-1\right\}$. (iii) $\frac{4}{3}$.

48. 126. 49. $a^8-16a^7x+112a^6x^2-448a^5x^3+....$

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V.

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6. $x+\frac{x}{y}-1$. 7. 6; 9.
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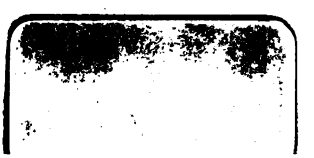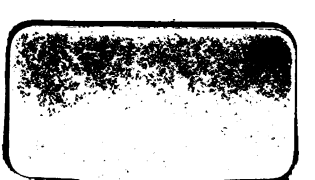
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